Faculty of Natural and Applied Sciences Journal of Applied and Physical Sciences
Print ISSN: 3026-9857
www.fnasjournals.com
Volume 2; Issue 2; March 2025; Page No. 1-11.

On some applications of Semigroups

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Abstract
This paper examines application of semigroup theory. Semigroup theory is a branch of algebra that involves three elements with an associative operation and has wide applications. The paper examines the applications of semigroups in the areas of research, probability theory, partial differential equation, biology, quantum mechanics, sociology, linguistics, computer science, games and in humanity.

Keywords: Application, Applied Mathematics, Humanity, Research, Semigroup Theory

Introduction

Semigroup is a fundamental concept in modern algebra. The concept is a generalization of the concept of a group whereby half of the group axioms remains; hence the term “semigroup”. Semigroup theory has evolved over the years and more discoveries are being recorded (Abubakar, 2021).The concept of a semigroup has gained grounds over the years after fragmentary studies were carried out early in the twentieth century. Then the necessity of studying general transformations, rather than only invertible transformations (which played a large role in the development of group theory) became clear. Semigroup carry a natural involution that is, a mapping

$$S \to S \ni (st)^{-} = t^{-}s^{-} \text{ and } (s^{-})^{-} = s \text{ (group involution is } s^{-} = s^{-1}).$$

Iqra and Younas (2021) posited that existence of a neutral element e is not automatic in semigroups. Howie (1986) posited that one of the pure mathematical reason to study semigroup is that semigroup is fun , providing an elegant theory with arguments that any Mathematician can actively enjoy for instance regularity :

$(S, .)$ is regular if $\forall a \in S, \exists\, x \in S$ such that

$$axa = a$$

ax and xa are idempotent . $(S, .)$ is regular iff $\forall a \in S, \exists\, a' \in S$

$$aa'a = a, a'aa' = a'$$

Usually , a' is called inverse element , inverse element here is a weaker concept than in groups (Howie, 2003).

Semigroups have been a useful tool in research, discovering new concepts in semigroups like partial right congruences (Ndubuisi et al., 2023); new concepts in algebra like in paralletrix rings (Ndubuisi et al., 2024) new methods of generating more structural examples of semigroups (Abubakar, et al., 2020), new computational techniques (Abubakar et al., 2021), connection between semigroups as in homomorphism between J*simple semigroups (Ndubuisi et al., 2019) in other areas of mathematics like in graph theory (Abubakar & David, 2015) in probability theory (Iqra & Younas, 2021).

In Sociology , semigroups are very useful and relevant in the area of kinships and relationships (Reddy &Dawud, 2015) . The theory of one-parameter operator semigroups serve as a powerful tool in the study of qualitative and quantitative properties of solutions (Nagel & Rhandi , 2020) . Partial differential equations, a useful tool in applied mathematics also used semigroup to investigate connections , focusing on their applications in mathematical analysis and physics especially in the area of time dependent equations as an ordinary differential equation on a function space (Iqra &Younas. 2021).

Cite this article as:
Abubakar, R. B. (2025). On some applications of Semigroups. *FNAS Journal of Applied and Physical Sciences*, 2(2), 1-11.

One reason for discussing and studying semigroups is that they have several applications. For instance, in biology they are being used for classifying organisms vis-à-vis the hereditary laws (Reddy & Dawud, 2015). They are also useful for studying the DNA protein-coding problem. Interestingly, semigroups are also being used in some of the social sciences to study various aspects of social and financial networks. Semigroups have also been applied to games , a legendry example is the misère games (Plambeck &Siegel, 2013). During the past few decades connection in the theory of semigroups and the theory of machines have become of increasing importance, both theories enriching each other. In association with the study of machines and automata, other areas of applications such as formal languages and the software uses the language of modern algebra in terms of Boolean algebra have been discovered (Zenab, 2014).

There is a close relationship that semigroups have with applications pertaining to automata and formal languages in Linguistics where free semigroups have been used to explain the concept of grammar and its relevance to Markov 0-simple semigroups (Omoelebe &David, 2015). The concept of kernels using restriction semigroups have also been discovered (Abubakar, 2021). Semigroups arise in so many places bridging between disciplines : Semigroups bridges between partial differential equations and probability theory . In functional analysis, semigroups are isomorphic to the non negative reals. With all these reasons on why there is the need to study semigroups.

Semigroups

A semigroup is an ordered pair (S, *) where S is a non-empty set and $*$ is an associative binary operation on S. That is, $\forall x, y, z \in S$

$$((x * y) * z) = (x * (y * z))$$

Usually, we sometimes denote the semigroup (S, *) by only S, if the underlying operation is understood. Examples of semigroups include : $(\mathbb{Z},+),(\mathbb{R},\times),(\mathbb{Q},+),(\mathbb{C},+)$, every non-empty set can be turned into a semigroup. Usually since the operation " $-$ subtraction" and '$\div$ division" are not readily associative, semigroups usually with the operations of $-$ and $\div$ are not usually common.

Semigroups constitute the simple , most manageable and most natural class of algebras to which the methods of Universal algebra must be applied (Howie, 1986). Semigroup is a non-addictive analogue of ring theory Gould & Lawson, 2019) . There are several types of semigroups which includes :

i.Null semigroups:-Let S be a non empty set and '0' $\in S$, we define a null semigroup by decreeing $xy = 0, \forall x, y \in S$

ii.Orthodox Semigroups:-A semigroup S is called Orthodox if it is regular and the idempotents form a subsemigroup examples include Inverse semigroups, band etc

iii.Commutative Semigroups:-If a semigroup operation is commutative, then the semigroup is called a commutative or abelian semigroup

iv.C-Semigroups:-A semigroup with an additional unary operation $C \ni$

$$aCa = a, C(a)C(b) = C(b)C(a);\ C(C(a) = C(a), CC(b) = C(ab)\,.$$

It is sometimes called 'closure semigroups'

v.Free Semigroups :-Let X be an arbitrary set and let the operation * be defined on the set F_x of all finite sequence of elements of X by means of the formula $(x_1, . . . , x_n)\ \ (y_1 , . . . , y_n) = (x_1y_1, . . ., x_ny_n)$.
$F_x$is called a free semigroup on X

vi.Monogenic Semigroups:-Let S be a semigroup , and let $U_i = i \in I/I \neq \emptyset\}$be an indexed family of subsemigroup of S .The intersection U of all subsemigroups U_i, if not empty is again a subsemigroup of S.for every non empty subset A of S, there is at least one subsemigroup of S containing A , namely S itself..Hence, the intersection of all subsemigroups of S containing A is a subsemigroup of S containing A denoted by $\langle A\rangle$. Hence $\langle a\rangle = \{a, a^2, a^3, . . . \}$. The semigroup S=$\langle a\rangle$ is called a Monogenic semigroup

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vii.Symmetric Semigroups:-If a set of transformation of an arbitrary set M is closed under the operation of composition, it is closed with respect to this operation and such a semigroup is the set of all transformation of M called a Symmetric semigroup on M.

viii.Group bound Semigroups :-A group bound semigroup S is a semigroup with the property that every element a has a power a^n ($n \geq 1$) lieing in a subgroup of S.

ix.U-semigroups:-A semigroup (S, μ) is called a U-semigroup if a unary operation $a \rightarrow a'$ is defined on S with the property that $(a')' = a, \forall a \in S$. (*-semigroup $(a^*)^* = a, \forall a \in S$; I-semigroup $(a^{-1})^{-1} = a, \forall a \in S$)

x.W-semigroups:-A semigroup whose semilattice of idempotents are isomorphic to C_W

xi.Inverse semigroups:-A semigroup in which every element has an inverse in the sense of $xyx = x$ and $yxy = y$

x.Nilpotent semigroups:-A semigroup S is called a nilpotent semigroup if $|S^r|$ =1 , for some $r \in \mathbb{N}$

xi.M-semigroups:-A semigroup S is an M-semigroup if E(S) is an M-biordered set

xii.0-Simple semigroups:-A 0-simple semigroup is a semigroup that has only two ideals , itself and $\{0\}$ and

$S^2 \neq \{0\}$.

xiii.Zero semigroups:-A semigroup S in which $\forall x, y \in S, xy = y \text{ and } yx = x$.

xiv.Regular semigroups :-A semigroup in which each element is regular is called a Regular semigroup

xv.R-Unipotent semigroups:-An inverse semigroup in which the idempotents form a semilattice is called a R-unipotent semigroup (equivalently R-regular semigroups in which the idempotents forms a right normal band)

xvi.Completely semisimple semigroups :-A regular semigroup S is called a completely semisimple semigroup if no pair of distinct comparable elements are D −related

xvii.T-semigroups :-If a semigroup S has a congruence σ such that $S/\sigma$is a band and each $\sigma - class$ is a semigroup of type T, S is called a band of type T-semigroup

xviii. P-semigroups:-An E-unitary inverse semigroup where E is the semilattice of S and σ its least group congruence($S/\sigma \approx G.S$) is regarded as a P-semigroup where P is a triple (G, E, g)

xix.Reductive semigroup:-A semigroup that is right reductive that is $xa = xb$ (right reductive) and $ax = bx$ (left reductive) is called a Reductive semigroup

xx.Adequate semigroups:-A semigroup S is left adequate if idempotents commute and for each $a \in S$, there is an idempotent $e \in S \ni xa = ya \Leftrightarrow ex = ey$.

xxi.F-inverse semigroup :-An inverse semigroup S is said to be an F-inverse semigroup if f every class contains a greatest element , S has an identity element and E is unitary

xxii.Topological semigroup :-A topological semigroup consists of a semigroup endowed with a Hausdorff topology such that the multiplication function $(s, t) \longmapsto st: S \times S$ is jointly continuous

xxiii.Restriction semigroups:- Restriction semigroups are semigroups with an additional unary operation $^+$ (left restriction) and * (right restriction)

xxiv.Munn Semigroups:-Let E be a semilattice , for each e$\in E$, the set $E_e = \{i \in E : i \leq e\}$is a principal ideal of E.$T_E = \cup \{T_e : (e, f) \in U\}$ is called the Munn semigroup of the semilattice E

xxv.Tzeitin's semigroup:The presentation S $= \langle a, b, c, d, e \mid ac = ca, ad = da, bd = db, eca = ce, edb = de, cca = ccae \rangle$ is called the Tzeitin's semigroup. This semigroup has an undecidable word problem that is, there exists no algorithm m that determines whether u =v is a relation in this semigroup where $u, v \in \{a, b, c, d, e\}^+$ are given (input) words.

xxvi.Clifford semigroups :-A Clifford semigroup is defined as a complete semigroup with a triple $(S,\ \mu,\ ^{-1})$ $in\ which\ \forall x, y\ \in S\ \ (xx)(yy^{-1}) = (yy)(xx)$. It is both an inverse and a completely regular semigroup.

xxvii.Rees semigroups:-Let G be a group with identity element e and let I, ∧ be non-empty sets .Let P = $(P_{\lambda i})$ be a $\wedge \times I$ matrix with entries in the 0-group, suppose that P is regular in the sense that no row or column of P consists entirely of zeros then, $M^0[G; I;\wedge ; P]$ is called the Rees matrix semigroup

xxviii.Baer-Levi Semigroups (Foulis Semigroups):- A Baer-Levi Semigroup is a *-semigroup with 2 sided zero in which the right annihilator of every element coincides with the right ideal of some projection which can be expressed formally as $\forall x \in S\exists$ a particular projection $e\ \ni \{y \in S \mid xy = 0| = eS$.

xxixBicyclic semigroups:-This is a free semigroup on two generators p and q under the relation $pq = 1$, that is, each semigroup element is a string of those two letters, with the provision that 'p q' does not appear. Semigroup operation is a concaternation of strings which is clearly associative.

xxx. Brandts semigroups:- A completely simple semigroup and an inverse semigroup . For some group G and some index set T, the semigroup $M^0[G; I; I\ ; \Delta]$ is called a Brandts semigroup.

Others are Putcha semigroups, semigroup of order 2, BD-semigroups, Transformation semigroups, U-class semigroups, Abundant semigroups, weakly –ample semigroups , Nulloid semigroups, Archimedean semigroups, Cancellative semigroups, Trivial semigroups, Absolutely closed semigroups etc (Howie, 2003).

Some applications of semigroups

Applications of semigroups will be examined under four categories :

I. Research in Semigroups
II.Applied Mathematics
III.Other Fields
IV.Humanity

I.Research in Semigroups

a.Ndubuisi et al.(2023) studied characteristics of w-cosets of rhotrix type A subsemigroup which helps in studying rhotrix type A semigroups and considered some partial right congruences on rhotrix type A semigroups

b.Congruences and congruence classes in Semigroups (Abubakar & Asibong-ibe , 2018)

$PT_{\{1,2,3,4,5,6\}}$,$S = \{\,A, B, C, D, E\}$ 5 elements

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & x & 5 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & x & 3 & x & 5 & 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ x & x & 3 & x & 5 & 4 \end{pmatrix},$$
$$D = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ x & x & 3 & x & 5 & x \end{pmatrix},\quad E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ x & x & x & x & x & x \end{pmatrix}$$

$\wp\boldsymbol{T}_{\{1,2,3,4,5,6\}}$: **(2 4 3 $\boldsymbol{x}$ 5 1)**

+	A	B	C	D	E
A	B	C	D	D	E
B	C	D	D	D	E
C	D	D	C	D	E
D	D	D	D	D	E
E	E	E	E	E	E

$E_s\ = \{C, D, E\}$

The $\widetilde{R_s}$ –classes are : $\widetilde{R_D} = \{\,A, B, C\}$ and $\widetilde{R_E} = \{\,A, B, C, D\}$

$PT_{\{1,2,3,4\}}$ $S = \{A, B, C, D, F, G, H, I, J, K, L, M, N, O, P, E\}$ 16 elements

$$\wp \boldsymbol{T}_{\{1,2,3,4,\}}: (\mathbf{1}\ \mathbf{4}\ \boldsymbol{x}\ \boldsymbol{x})$$

+	A	B	C	D	F	G	H	I	J	K	L	M	N	O	P	E
A	B	B	E	D	F	G	E	E	E	I	F	E	D	E	F	E
B	B	B	E	D	F	G	E	E	E	E	F	E	D	E	F	E
C	E	E	E	E	E	E	E	E	I	E	E	E	E	C	I	E
D	E	E	E	E	E	E	E	E	E	D	E	E	E	E	E	E
F	E	E	E	E	E	E	E	E	D	E	E	E	E	F	D	E
G	D	E	F	E	E	E	B	D	E	E	D	G	F	E	E	E
H	H	H	E	I	C	M	E	E	E	E	C	E	I	E	C	E
I	E	E	E	E	E	E	E	E	E	I	E	E	E	E	E	E
J	E	E	E	E	E	E	E	E	E	J	E	E	E	E	E	E
K	E	E	E	E	E	E	E	E	E	K	E	E	E	E	E	E
L	E	E	E	E	E	E	E	E	D	I	E	E	E	F	D	E
M	I	E	C	E	E	E	H	I	E	E	I	M	C	E	E	E
N	E	E	E	E	E	E	E	E	I	D	E	E	E	C	I	E
O	E	E	E	E	E	E	E	E	J	E	E	E	E	O	J	E
P	E	E	E	E	E	E	E	E	D	J	E	E	E	F	D	E
E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E

$E_s = \{B, O, E\}$

The $\widetilde{R_s}$ –classes are : $\widetilde{R_D} = \widetilde{R_F} = \widetilde{R_G} = \{A, B\}, \widetilde{R_E} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P\}$

where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & x & x \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & x & x & x \end{pmatrix}, C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & 3 & x & x \end{pmatrix}, D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & x & x & x \end{pmatrix},$$
$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & x & x & x \end{pmatrix},\ G = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & x & x & x \end{pmatrix},\ H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & 1 & x & x \end{pmatrix}, I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & 4 & x & x \end{pmatrix},$$
$$J = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & x & 4 & x \end{pmatrix},\ K = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & x & x & 4 \end{pmatrix},\ L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & x & x \end{pmatrix}, M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & 2 & x & x \end{pmatrix},$$
$$N = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & x & x \end{pmatrix},\ O = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & x & 3 & x \end{pmatrix},\ P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & x & 4 & x \end{pmatrix}\ E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ x & x & x & x \end{pmatrix}$$

c. Congruences and kernels: Abubakar (2022) presented some left restriction semigroups LRS in partial transformation $\wp\mathfrak{I}_{\{X=11,12.13.14.15\}}$, computed the left congruences $\tilde{R}$-classes and enumerating all the kernels $K(\wp\mathfrak{I}_{\{11,12,13,14,15\}})$ for each element of LRS in $\wp\mathfrak{I}_X$. Left restriction semigroups have appeared at the convergence of several flow of research, they model unary semigroups of partial mappings on a set where the unary operation takes a map to the identity map on its domain. Partial transformation semigroup $\wp\mathfrak{I}_X$ is a weakly left E-ample semigroup otherwise known as Left restriction semigroup.The identities that define a Left restriction semigroup S are

$$a^+ a = a, \qquad a^+ b^+ = b^+ a^+, (a^+ b)^+ = a^+ b^+, ab^+ = (ab)^+ a$$
$$E = \{a^+ : a \in S\}$$

E is a semilattice known as the semilattice of projections of S (Zenab, 2014). Let $\wp T_X$ be a Left restriction semigroup with distinguished semilattice,

$$E = \{I_Y : Y \subseteq X\}$$

and with

$$\alpha^+ = 1_{dom\ \alpha}$$

Restriction semigroups are a generalization of inverse Semigroups. Restriction semigroups are semigroups with an additional unary operation $^+$ (Left) and $*$ (Right). A semigroup S is left restriction with respect to $E \subseteq E(S)$ if `

i. E is a subsemilattice of S
ii. Every element $a \in S$ is $\widetilde{R_E}$ -related to an element of E (denoted by a^+)
iii. $\widetilde{R_E}$ is a left congruence

iv. The left ample condition holds $\forall\, a \in S \; and \; e \in E,$

$$ae = (ae)^{+}a$$

Computing the congruence class $-\tilde{R}$ -classes and the kernels

For $\wp\mathfrak{I}_{\{X=11\}}$ $\wp\boldsymbol{T}_{\{1,2,3,4,5,6,7,8,9,10,11\}}$: $(1\ 3\ 6\ 7\ 8\ X\ 9\ X\ 10\ X\ X)$

+	A	B	C	D	F	G	H	I	J	K	L	M	N	O	E
A	B	C	D	D	E	G	H	I	J	K	L	M	N	O	E
B	C	D	D	D	E	G	H	I	J	K	L	M	N	O	E
C	D	D	D	D	E	G	H	I	J	K	L	M	N	O	E
D	D	D	D	D	E	G	H	I	J	K	L	M	N	O	E
F	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
G	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
H	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
I	H	E	E	E	E	E	E	E	E	E	E	E	E	E	E
J	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
K	I	H	E	E	E	E	E	E	E	E	E	E	E	E	E
L	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
M	J	E	E	E	E	E	E	E	E	E	E	E	E	E	E
N	K	I	H	E	E	E	E	E	E	E	E	E	E	E	E
O	L	E	E	E	E	E	E	E	E	E	E	E	E	E	E
E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E

$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 3 & 6 & 7 & 8 & X & 9 & X & 10 & X & X \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 6 & X & 9 & X & X & 10 & X & X & X & X \end{pmatrix},$

$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & X & X & 10 & X & X & X & X & X & X & X \end{pmatrix}, D = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ X & 10 & X & X & X & X & X & X & X & X & X \end{pmatrix}, G = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$H = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & X & X & X & X & X & X & X & X & X & X \end{pmatrix}, I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$J = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & X & X & X & X & X & X & X & X & X & X \end{pmatrix}, K = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & X & X & X & X & X & X & X & X & X & X \end{pmatrix}, M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$N = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & X & X & X & X & X & X & X & X & X & X \end{pmatrix}, O = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ X & X & X & X & X & X & X & X & X & X & X \end{pmatrix},$$

Left congruence $\tilde{R} - classes$

$\widetilde{R_D} = \widetilde{R_G} = \widetilde{R_H} = \widetilde{R_I} = \widetilde{R_J} = \widetilde{R_K} = \widetilde{R_L} = \widetilde{R_M} = \widetilde{R_N} = \widetilde{R_O} = \{A, B, C, D\},$

$\widetilde{R_E} = \{A, B, C, D, F, G, H, I, J, K, L, N, O, E\}$

Kernels

$K_A = \{C, D\}_D ; \{F, G, H, J, L, E\}_E ; K_B = \{B, C, D\}_D ; \{F, G, H, J, K, L, M, O, E\}_E;$

$K_C = \{A, B, C, D\}_D ; \{F, G, H, J, K, L, M, O, E\}_E ; K_D = \{A, B, C, D\}_D ; \{F, G, H, J, K, L, M, N, O, E\}_E$

$; K_G = \{A, B, C, D\}_G ; \{F, G, H, J, K, L, M, N, O, E\}_E ;$ $K_H \{A, B, C, D\}_H ; \{F, G, H, J, K, L, M, N, O, E\}_E$

$K_I = \{A, B, C, D\}_I ; \{F, G, H, J, K, L, M, N, O, E\}_E ; K_J = \{A, B, C, D\}_J ; \{F, G, H, J, K, L, M, N, O, E\}_E$

$; K_K = \{A, B, C, D\}_K ; \{F, G, H, J, K, L, M, N, O, E\}_E ; K_L = \{A, B, C, D\}_L ; \{F, G, H, J, K, L, M, N, O, E\}_E$

$; K_M = \{A, B, C, D\}_M ; \{F, G, H, J, K, L, M, N, O, E\}_E ; K_N \{A, B, C, D\}_N ; \{F, G, H, J, K, L, M, N, O, E\}_E$

$K_0 = \{A, B, C, D\}_O ; \{F, G, H, J, K, L, M, N, O, E\}_E ; K_E = K_F = \{A, B, C, D, F, G, H, J, K, L, M, N, O, E\}_E$

(Abubakar, 2022)

II. **Applied Mathematics**

a.Semigroups in probability theory

One fundamental operation in probability theory is convolution and a larger part of classical probability theory deals with $M'_{+}(\mathbb{R})$ partial derivative on $\mathbb{R}$ being a (commutative) semigroup with respect to convolution and carrying the natural involution u^{-}

$$(B) = \mu(-B)$$

Two basic procedures of obtaining new distributions from given ones are finite convolution products and limits of such finite convolutions (in the weak topology).Both of these operations are in general difficult to perform by direct calculation as an aid , we have the classical fourier transformation

$$F: M'_{+}(\mathbb{R}) \rightarrow C^{\mathbb{R}}$$

$$\mu \mapsto (t \longmapsto \int e^{itx}\, d\mu(x)\ )$$

being a semigroup isomorphism and also a homoemorphism onto its image if we consider $C^{\mathbb{R}}$ with point wise operations (Iqra & Younas, 2021).

b. Semigroup in graph theory

Abubakar and David (2015) constructed a graph using 0-simple Semigroup S. S is a semigroup that has no proper ideal except itself and zero. The use of graph has become widespread in the algebraic theory of semi groups. The graph is mainly used as a visual, to make presentations clearer and the problems more imaginable. Central to this approach is the cayley graph of semigroups. Cayley graph is a graph that encodes the structure of a semigroup. It uses a specified, usually finite set of generators. It is a method of presentation of a group where the vertices of the graph are the elements of the semigroup called the generators of the semigroup. The technique of using Cayley graph of a group presentation for construction is called Graph Expansion.

They considered a 0-simple semigroup presentation (X, f, S) where X is finite , non-empty set, S, a 0-simple semigroup and f: X→S, f is a function such that Xf generates S as a 0simple semigroup. Here, VG= S and ($e1$)= 0, if ei= 0 for only i =1, $t(ei)$ = s if s ∈ S. Hence the edge set

E_G= { (s, x, s(xf) : s ∈ S and x ∈ X}

where s ● ————— ● s (xf)

$i(s, x, s(xf)) = 0$ if $s = 0$
$i(s, x, s(xf)) = s$ if $s \neq 0$
$t(s, x, s(xf)) = s(xf)$ if $s \in S$

Let 0-simple semigroup S act on G where s ∈ S and (s, x, s(xf)) ∈E_G we have to note that for (t, x, t (xf)) ∈ E_G , s ∈S $s(t, x, t(xf)) = (st, x, st(xf))$

Considering Δ_G,the subgraph of G, the graph expansion is defined thus: M = M (X, f, S) of (X, f ,S) a 0-simple semigroup presentation is expressed thus M = { (Δ_G,s) : Δ_G is a finite 0-rooted subgraph of G and 0, s ∈ V_{Δ_G}}. We now define a multiplication on M by

(Δ_G, s) ($\sum_G$, t) = {($\Delta_G \cup s\sum_G$,) s, ≠ 0 and s.t ≠ 0 (Δ_G,0) if s or t = 0,$s.t$ = 0

(Δ_G,) 0 = 0 (Δ_G,) = (Δ_G,0)= 0, a∈ Δ_G We identify all the element of the form (Δ_G,0) with 0 under the multiplication given (Abubakar & David, 2015).

c.Semigroups in Partial differential equations

Semigroup theory can be used to study some problems in the field of partial differential equations. Semigroup approach is to regard a time-dependent partial differential equation as an ordinary differential equation on a function space. Example

Consider the following initial boundary value problem for the heat equation on the spatial interval (0,1) ⊂ $\mathbb{R}$ and time times $t \geq 0$

$$\begin{cases} \partial tu(t,x) = \partial xu(t,x) \;\; x \in (0,1), t > 0 \\ \mu(t,x) = 0, x \in \{0,1\}, t > 0 \\ u(t,x) = u(x), x \in (0,1), t = 0 \end{cases}$$

Cite this article as:
Abubakar, R. B. (2025). On some applications of Semigroups. *FNAS Journal of Applied and Physical Sciences*, 2(2), 1-11.

Let $X = L^2(0,1)R)$ be the L^P space of square –integrable ℝ function with domain the interval $(0,1)$.Let A be the second derivative operator with domain

$$D(A) = \{u \in \ H^2\ ((0,1); \mathbb{R}/u(0) = u(1) = 0\}$$

Where H^2 is the solvable space .Then, the above ibvp can be interpreted as an ivp for an ordinary differential equation on the space X as

$$\begin{cases} u(t) = Au(t) \\ \ \ u(0) = u \end{cases}$$

On a heuristic level, the solution to this problem "ought" to be $u(t) = \exp(tA)u_0$.However, for a rigorous treatment, a meaning must be given to the exp of tA.As a function of t, $\exp(tA)$ is a semigroup of operators from x to itself , taking the initial state $u(t) = \exp(tA)\, u_0$ at time t. The operator A is said to be infinitesimal (Iqra &Younas, 2021).

III. Semigroup application in other fields

a.In Biology

Semigroups are used in biology to describe certain aspects in the crossing of organisms, in genetics and in consideration of metabolisms

Example

In breeding a strain of cattle , which can be black or brown monochromatics or spotted , it is known that black is dominant and brown receive and that monochromatic is dominant over spotted. Thus there are four possible types of cattle in this herd.

$a - Black\ monochromatic$ $\qquad c - brown\ monochromatic$

$b - black\ spotted$ $\qquad d - brown\ spotted$

Due to dominance, in crossing a black spotted one with a monochromatic one, we expect a black mono chromatic one, we expect a black monochromatic .This can be symbolized by $bc = a$

The Cayley table is best representative of this

*	a	b	c	d
a	a	a	a	a
b	a	b	a	B
c	a	a	c	C
d	a	b	c	D

The $S = \{\,a, b, c, d\}$ is a semigroup with the identity element d (Reddy & Dawud,2015).

b. Quantum mechnanics : Baer semigroups are encountered in quantum mechanics in particular as the multiplicative semigroups of Baer*-rings. If H is a Hilbert space, then the multiplicative semigroup of all bounded operators on H is a Baer *- Semigroup. The involution, in this case maps an operator to its adjoint.

c. Semigroup in Sociology

With the human interactive behavior in societal structure , the relational study can be formulated in language of semigroups.

Defining a relation relational semigroup as follows:

Definition : $(R(M),\ \circ\ )$ is called a relational semigroup on M where $\circ$ is the relational product , obviously $R \circ R \subseteq R$. Defining a kinship system. A kinship system is a semigroup $S = \{X, R\}$ where R is a relation on X which express equality of kinship relations.

Example

i. Let $X = \{$ is father of, is mother of $\}$ and $\circ R = \emptyset$.The kinship system S is the semigroup $\{$is father of, is mother of, is grafather of on father's side of $\}$

ii. Let$F =$ is father of " , $M =$ "is mother of" , $S =$ "is son of ",$D =$ is daughter of"

then $X = F, M, S, D, B, Si, C$ " and $R =$

$\{\,(CM, CF), (BS, S), (S, D, D), (CBM, CMM), (MC, FM), (SB, S), (DS, D), (MBC, MMC), \ldots\}$

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the first means that in the semigroup, we have $CM = CF$, children of Mother are the same as the children of the father .
G −Society(non-empty set of people), $S(G)$semigroup of different kinship relations of this society . Semigroup $S(G)$ often have special properties example "is son of" and " is father of" are nearly inverse relations.

iii.Let the kinship system $S = \{X, R\}$

$$X = \{p = \text{(is parent of)}, C = \text{(is child of)}\}$$

$$R = \{(PP, P), (PC, CP), (CC, C)\}$$

i.Let a, b, c be the equivalence classes of P, C and PC respectively. Now " ∘ " is given by

∘	a	b	c
A	a	c	c
B	c	b	c
C	c	c	c

(S, ∘) is a Semigroup

iv..Let Si =Sister and $D = daughter$ then

$X = -Si, D$",

$$R = \{-(SiD, D), (DSi, D), (SiSi, Si), (DD, D)\}$$

Then
$S = -[Si], [D]$is regular semigroup but not inverse semigroup with the following operation table

∘	Si	D
Si	Si	D
D	D	Si

(Reddy &Dawud, 2015).

d.Linguistics : formal languages

An automaton is a machine through which input data transforms into some output. Examples are : telephone switch board, computer, etc. Languages are spoken all over the globe in different clime .In Nigeria, citizens are familiar with two or three languages that are spoken around : like English, Ikwerre, Hausa, Ibo , Yoruba, Ogbia, afemai etc . These are examples of 'natural' languages. Apart from such languages, mathematicians and computer scientists have defined 'formal' languages. These languages also have alphabets and words, but the words may not mean anything to a listener or reader. Such languages focus on the structure of the statements or elements, that is, the syntax. Thus, they are useful for studying linguistic patterns as well as the syntax of programming languages. We have earlier defined a Free semigroup, lets put it in perspective now.

Word
A word on (x, y) is a finte string of the symbols x, x^{-1}, y, y^{-1} where $x\ and\ x^{-1}$ cannot be adjacent because together, they form the empty word denoted by I,$\wedge$ or $\{0\}$.

Example let A={a, b}* then, baabbaba ∈ A^8, $A^0 = \{\wedge\}$

Any spoken statement , written statement ,any input to a computer is a string of symbols just like every Shakespeare play is an element of a non empty set where A is the set consisting of the 52 –upper and lower case letters of the alphabet together with various space and punctuations.

If A is an alphabet, A* is the set of all strings obtained by concatenation zero or more symbols from A, and A* = A+ - {Λ}. If A = {a, b} then A* = {Λ, a, b, ab, aa, ….}. In general a language is defined as a subset of A*.

Gram m a r: A model to describe language is known as grammar. There are essentially three ways to construct a language.
a) Approach via grammar
b)Approach via automata
c)Algebraic approach (Nagel &Rhandi, 2020)

Let A be a non-empty set. Let * A be the free semigroup on A, i.e., ∗ A is F { }, A ∪ Λ where Λ is the empty word. So * A is the set of all words over A, i.e., A {a a a a A, . . . {0}}, together with the operation of concatenation (or juxtaposition). A formal language L over the set A is a subset of A. * Here A is called the alphabet over which L is defined, and elements of * A are called words.

For example, L can be all of A* this is called the universal language. Another example is L , «= which is called the empty language. So, as you can see from the above, a formal language is about the form (i.e., the syntax) and not about meaning. Further, the mathematical theory of formal languages doesn't study individual languages, but the classes of language, and the mechanisms that describe these classes. These are closely linked with automata.

Automata is the study of different mechanisms for generation and recognition of languages. Noam Chomsky, American linguist and cognitive scientist, has presented an hierarchy of these classes
i. Regular languages, characterised by finite state automata; L :
ii. Context-free languages, characterised by pushdown automata; L :
iii. Context-sensitive languages, characterised by linear bounded automata; L :
iv. Computable languages, characterised by Turing machines.
The semigroups that characterize these classes of languages are the Free semigroups corresponding to the automaton concerned. In which , concatenation, coding(error-correcting, error-detecting, prefix, biprefix, suffix etc),isomorphism determined by the cardinality of the alphabet called the rank of free semigroups etc Application of this is in Marriage Automaton (Reddy & Dawud, 2015).

e.Computer Science :
Applications of partial transformations semigroups PT_X include : Rail track lining (in exchange programme), traffic flow algorithms, fluid mechanics , movement algorithm and artificial intelligence for digitalized change to create intriguing patterns.
Consider $f: G \to H$

$$\text{Ker } f = \{g \in \ G/\ f(g) = e_H\}$$

Semigroups do not have identities hence, their kernels are all elements mapped to the same elements .As kernel is a process manager , a foundational layer of operating system, managing hardware resources like RAM and CPU and applications like running processes managing hard disks and interrupts are processed faster with the kernels. The kernels computed in Abubakar (2022) from Left restriction semigroups form a whole new foundational layer of operating systems .

f. Games: The legendry Misere games
Commutative Semigroups and their analysis show up in the theory of Misere combinatorial games.

The "misère quotient' semigroup construction gives a natural generalization of the normal-play spraue-Grundy theory to misère play which allows for complete analysis of many of such games https://miseregames.org
Examples i. Lattice games without rational strategies (4th September 2012)
ii.The secrets of Notakto : winning at X-only .Tic-Tac-Toe (8th Jan, 2013)
iii.Structure and classification of Misere quotients (4th March, 2007)
iv.Advances in loosing(Plambeck & Siegel 2013)

IV.Humanity

Saving the best for the last . Semigroup is an associative groupoid, this property of associativity distinctively describes any semigroup .The power of Associativity cannot be underestimated in human association. I know you, you know me, the three of us know each other , its super.

$$\forall\, x, y, z \in R$$
$$(x * y) * z = x * (y * z)$$

Any gathering can be viewed as a Semigroup because there is an associative nature among members of the gathering and this fosters growth and connectedness boosting humanity which is what bounds us all in the universe on earth.

Hence,

$$(\boldsymbol{Gathering},\ *) \text{ is a Semigroup}$$

where the gathering has a minimum of three members ,which is obvious by the relationship of association that exists. Any gathering of a minimum of three members will be considered as a semigroup. We cannot underestimate the power of associativity in any gathering , it forms a great bond that moves humanity forward. The type of associativity defined and followed in the gathering can take the form of any of the type of semigroup already discussed.

Conclusion

Semigroup constitute the simplest, most manageable and most natural class of algebra to which the methods of Universal algebra must be applied (Howie, 2003). Semigroup is a non-addictive analogue of ring theory (Gould &Lawson, 2019). Conclusively, Semigroup family in its purest form, in research and applications and in virtually all fields is unique and interesting , more types of semigroups are being discovered with fascinating intrigues in this interesting mathematical world that will have diverse applications.

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