FNAS Journal of Mathematical Modeling and Numerical Simulation
Print ISSN: 3027-1282
www.fnasjournals.com
Volume 2; Issue 4; December 2025; Page No. 45-50.
DOI: https://doi.org/10.63561/jmns.v2i4.1122

A Graph-Theoretic Characterisation of Generating Sets in Finite Full Transformation Semigroups

***[1]Lawal, Z., [2]Ali, B.., & [3]Abubakar, U.I.**
[1,3]Department of Mathematics, Federal University of Education, Zaria.
[2]Department of Mathematics, Faculty of Physical Sciences, Nigerian Defense Academy, Kaduna

***Corresponding author emai**l: lawalzaharadeen57@gmail.com,

Abstract
This paper investigates the structural characterization of generating sets within the semigroup of full transformations, emphasizing the graph-theoretic proper- ties of their associated digraphs. Focusing on the Jn−1-class, denoted by Jn−1 = {α ∈ Tn: | im(α)| = n − 1}, we establish necessary and sufficient conditions for a subset A ⊆ Dn−1 to be a generating set. It is proved that A generates Dn−1 if and only if A is a cover and the corresponding digraph ΓA is strongly connected. Consequently, minimal generating sets are precisely the minimal strongly connected covers of Dn−1. Furthermore, every generating subset induces a connected and cyclic digraph, thereby revealing intrinsic links between algebraic generation and graph connectivity. Illustrative examples for n = 4 and n = 5 are presented through egg-box diagrams and directed graphs, showing that six and ten elements, respectively, suffice to generate the semigroup. These values correspond to the Stirling numbers of the second kind, which determine the number of partitions required to cover the associated transformation graphs in Singn. The results provide a deeper understanding of the combinatorial and graph-theoretic structure of generating systems in finite full transformation semigroups

Keywords:Transformation, Semigroups, generating sets, Strongly connected, Digraphs.

Introduction

The study of generating sets in transformation semigroups has long been a central subject in semigroup theory, motivated by foundational results of Howie (1978) on idempotent generation and the structure of singular transformations. Recent approaches have enriched these algebraic investigations by introducing *graph- theoretic* viewpoints: associating digraphs to generating sets and exploring how connectivity properties of these graphs reflect algebraic generation in transformation semigroups. In" Lengths of words in transformation semigroups generated by di- graphs", Cameron et al. systematically develops this connection by constructing semigroups from simple digraphs and analyzing word lengths in generators derived from edges. Similarly, East et al (2019) treat how semigroup properties, such as regularity and D-classes, correspond to the structure of the generating digraph. These works suggest a deeper principle: *connectivity in the digraph ensures semigroup generation properties*. At the same time, more general semigroup generation questions—such as determination of smallest generating sets, rank, and ubiquity—have been explored in broader contexts. For example, Jonuˇsas and Troscheit (2017) investigate unique irredundant generating sets, probabilistic properties, and computational rank problems. Additionally, studies such as Lallement et al. (1980) examine generation by idempotents and minimal generating sets in transformation semigroups. Motivated by these advances, the present work introduces a digraph-based equivalence for generation in the class D_{n-1}, showing that for $A \subseteq D_{n-1}$, A is a generating set if and only if it is a cover of D_{n-1} and the corresponding digraph Γ_A is strongly connected. This approach leverages combinatorial, graph-theoretic, and algebraic ideas to establish necessary and sufficient conditions for generation, connecting them to minimal generating sets and minimal strongly connected covers. We illustrate the theory with explicit examples for small values of n, uncovering connections to the Stirling numbers of the second kind. Such results deepen our understanding of transformation semigroup generation and provide tools for exploring ranks, minimal generation, and digraph-semigroup correspondence in other algebraic contexts.

Preliminaries

Throughout this manuscript, we use the symbol Σ to denote a semigroup and

$\Gamma_n = \{1, 2, \ldots, n\}$

to represent a finite totally ordered set of size n. We record below several fundamental notions and conventions required later.

A *semigroup* is a non-empty set Σ endowed with a binary operation (often written by juxtaposition) that is associative; that is,

$$(xy)z = x(yz) \qquad \text{for all } x, y, z \in \Sigma.$$

If Σ contains an element ϵ such that $x\epsilon = \epsilon x = x$ for every $x \in \Sigma$, then Σ is called a *monoid*. An element $\sigma \in \Sigma$ satisfying $\sigma x = x\sigma = \sigma$ for all $x \in \Sigma$ is termed a *zero element*, and in this case, we say that Σ is a *semigroup with zero*. Whenever an identity or zero is absent, one may adjoin such an element formally; the resulting enlarged semigroups are written Σ^1 and Σ^0, respectively.

A non-empty subset $U \subseteq \Sigma$ is called a *subsemigroup* if it is closed under the operation of Σ, i.e.,

$$xy \in U \qquad \text{whenever } x, y \in U.$$

Let $\Gamma_n = \{1, 2, \ldots, n\}$. A mapping $\varphi : \Gamma_n \longrightarrow \Gamma_n$ is referred to as a *full transformation* of Γ_n. The collection of all full transformations,

$$T_n = \{\varphi : \Gamma_n \longrightarrow \Gamma_n\},$$

forms a semigroup under composition and is known as the *full transformation semi- group*. More generally, a map ψ is called a *partial transformation* of Γ_n if

$$\mathrm{Dom}(\psi) \subseteq \Gamma_n, \qquad \mathrm{Im}(\psi) \subseteq \Gamma_n.$$

The set of all partial transformations,

$$P_n = \{\psi : \mathrm{Dom}(\psi) \subseteq \Gamma_n\},$$

also forms a semigroup under composition and is called the *partial transformation semigroup*.

An element $a \in \Sigma$ is said to be *regular* (in the sense of von Neumann) if there exists $b \in \Sigma$ such that

$$aba = a.$$

The semigroup Σ is called *regular* if each of its elements is regular. A regular semigroup in which every element admits a unique inverse is an *inverse semigroup*.

For any semigroup Σ, Green's classical relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ are defined by

$$a \,\mathcal{L}\, b \iff \Sigma^1 a = \Sigma^1 b, \qquad a \,\mathcal{R}\, b \iff a\Sigma^1 = b\Sigma^1, \qquad a \,\mathcal{J}\, b \iff \Sigma^1 a \Sigma^1 = \Sigma^1 b \Sigma^1,$$

with

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \qquad \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

An element $e \in \Sigma$ is an *idempotent* if $e^2 = e$. The set of idempotents of Σ is denoted by $E(\Sigma)$. A semigroup in which all elements are idempotent is called a *band*. If Σ is generated by its idempotents, meaning $\langle E(\Sigma) \rangle = \Sigma$, then Σ is termed a *semi- band*.

If Σ possesses a zero-element $\mathbf{o}$, an element $\alpha \in \Sigma$ is *nilpotent* if $\alpha^m = \mathbf{o}$ for some $m \geq 1$, and the set of all such elements is denoted $N(\Sigma)$. An element $\beta \in \Sigma$ is a *quasi-idempotent* if

$$\beta \neq \beta^2 = \beta^4,$$

namely if its square is idempotent although β itself is not. Likewise, an element $\delta \in \Sigma$ is called *quasi-nilpotent* if $\delta^k = C$ for some $k \geq 1$, where C denotes a constant transformation.

The *rank* of a semigroup Σ is the size of the smallest generating set:

$$\mathrm{rank}(\Sigma) = \min\{|A| : A \subseteq \Sigma,\ \langle A \rangle = \Sigma\}.$$

Restricting the generating set to $E(\Sigma)$, $N(\Sigma)$, or the set of all Quasi-Idempotents yields the *idempotent rank*, *nilpotent rank*, and *quasi-idempotent rank*, respectively.

A generating set G of Σ is *minimal* if no proper subset of G still generates Σ. Similarly, a family of subsets $\{A_i\}$ is a *minimal cover* of a set X if $\bigcup_i A_i = X$ and no proper subfamily continue to cover X. These ideas are central to the study of ranks and generation in the semigroups investigated in this paper.

Cite this article as:
Lawal, Z., Ali, B., & Abubakar, U.I. (2025) A graph-theoretic characterisation of generating sets in finite full transformation semigroups. *FNAS Journal of Mathematical Modeling and Numerical Simulation*, *2*(4), 45-50. https://doi.org/10.63561/jmns.v2i4.1122

Results

The collection of full transformations, denoted by T_n, serves as a cornerstone in examining the structure of finite semigroups. semigroups. Within T_n, the **J** -classes, particularly $J_{n-1} = \{\alpha \in T_n : |\operatorname{im}(\alpha)| = n - 1\}$, provide a rich setting for exploring generating systems and their combinatorial properties. Understanding how subsets of D_{n-1} generate the semigroup reveals important relationships between algebraic generation and graph-theoretic connectivity.

In this section, we introduce a digraph Γ_A associated with each subset $A \subseteq D_{n-1}$, where vertices correspond to elements of D_{n-1} and directed edges describe the action of transformations in A. This graph-theoretic representation provides an intuitive and rigorous means of analyzing generating behavior in T_n.

The results that follow establish necessary and sufficient conditions under which a subset of D_{n-1} forms a generating set, characterize minimal generating sets in terms of strong connectivity, and illustrate these findings through explicit constructions for small values of n. The main theorems thus bridge semigroup generation with fundamental concepts from digraph theory and combinatorics.

Lemma 3.1. Let $A \subseteq D_{n-1}$. If A is a generating set, then Γ_A is connected (as an undirected graph).

Proof. A generates $D_{n-1} \Rightarrow$ for all $u, v \in D_{n-1}$, there exists a directed path $u \to v$ in Γ_A. Ignoring edge direction $\Rightarrow \Gamma_A$ is connected. □

Corollary 3.2. If $A \subseteq D_{n-1}$ is a generating set, then Γ_A contains at least one directed cycle.

Proof. Γ_A strongly connected $\Rightarrow \exists$ path $v \to v$ for some $v \in D_{n-1} \Rightarrow$ directed cycle exists. □

Lemma 3.3. Let $A \subseteq D_{n-1}$. If A is a cover but Γ_A is not strongly connected, then A is not generating.

Proof. ¬strong connectivity $\Rightarrow \exists\, u, v \in D_{n-1}$ such that no path $u \to v$ in $\Gamma_A \Rightarrow A$ does not generate D_{n-1}. □

Corollary 3.4. Minimal generating sets of D_{n-1} $\iff$ minimal strongly connected covers of D_{n-1}.

Proof. A minimal generating set $\Rightarrow A$ is a cover and Γ_A is strongly connected with minimal cardinality. □

Theorem 3.5. Let $A \subseteq D_{n-1}$. Then A is a generating set if and only if A is a cover of D_{n-1} and the associated digraph Γ_A is strongly connected.

Proof. Assume first that A is a generating set for D_{n-1}. Then for each $x \in D_{n-1}$, some map $a \in A$ satisfies $x \in \operatorname{dom}(a)$. Thus, A forms a cover of D_{n-1}. Furthermore, since A generates the entire structure, for any $p, q \in D_{n-1}$ there must exist a directed path in Γ_A linking p to q; hence the digraph Γ_A is strongly connected.

For the converse direction, suppose that A covers D_{n-1} and that Γ_A is strongly connected. Because each element of D_{n-1} lies in the domain of at least one member of A, and because any two vertices p, $q \in D_{n-1}$ can be joined by a directed path in Γ_A, every point of D_{n-1} can be reached via compositions of elements from A. Consequently, the action of A generates the entire semigroup D_{n-1}. □

To exemplify the process. Let $n = 4$ and consider the J_{n-1}-class i.e

$$J_3 = \{\alpha \in T_4 : |im\alpha| = 3\} \text{ Thus}$$

J_3 contains nC_r- L-classes and Stirling number

$$S\,(n, r) = S(n-1, r-1) + rs(n-1, r) \text{ −R-classes}$$

and the number of H-classes in

$J_3 = nC_r \cdot S\,(n, r) = S(n-1, r-1) + rs(n-1, r) = 24.$

Let the kernel classes be

$A_1 = |3, 4|$, $A_2 = |2, 4|$, $A_3 = |1, 4|$, $A_4 = |2, 3|$, $A_5 = |1, 3|$ and $A_6 = |1, 2|$ then the images be $X_n \setminus \{4\}$, $X_n \setminus \{3\}$, $X_n \setminus \{2\}$ and $X_n \setminus \{1\}$

Thus, we have the following egg-box picture for J_3

Cite this article as:
Lawal, Z., Ali, B., & Abubakar, U.I. (2025) A graph-theoretic characterisation of generating sets in finite full transformation semigroups. *FNAS Journal of Mathematical Modeling and Numerical Simulation, 2*(4), 45-50. https://doi.org/10.63561/jmns.v2i4.1122

$J_3(T_4)$	$X_n \setminus \{4\}$	$X_n \setminus \{3\}$	$X_n \setminus \{2\}$	$X_n \setminus \{1\}$
$\{3,4\}$	ε_1	ε_2	α_1	
$\{2,4\}$	$\varepsilon_3,\ \alpha_2$		ε_4	
$\{1,4\}$	ε_5			$\varepsilon_6,\ \alpha_3$
$\{2,3\}$		$\varepsilon_7,\ \alpha_4$	ε_8	
$\{1,3\}$		ε_9	α_5	$\varepsilon 10$
$\{1,2\}$			$\varepsilon 11$	$\varepsilon_{12},\ \alpha_6$

Table 1: Elements of $J_3(T_4)$ categorized by images missing a single element

Where:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3,4 \\ 1 & 3 & 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 24 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 14 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & 23 & 3 \\ 1 & 2 & 4 \end{pmatrix},$$

$$\alpha_4 = \begin{pmatrix} 1 & 23 & 3 \\ 1 & 2 & 4 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 13 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 12 & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix}$$

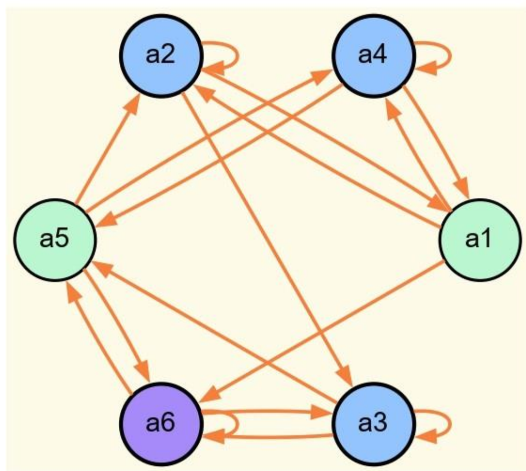

Figure 1: Directed graph of a Transformation Semigroup For n=4

For n=5, $J_{n-1} - class$ i.e

$J_4 = \{\alpha \in T_5 : |im\alpha| = 4\}$

$J_4(T_5)$	$X_n \setminus \{5\}$	$X_n \setminus \{4\}$	$X_n \setminus \{3\}$	$X_n \setminus \{2\}$	$X_n \setminus \{1\}$
$\{1,5\}$	$\alpha_1,\ \varepsilon_1$				ε_2
$\{2,5\}$	ε_3	α_2	ε_4		
$\{3,5\}$	ε_5		$\alpha_3,\ \varepsilon_6$		
$\{4,5\}$	ε_7	α_8		α_4	
$\{1,4\}$		ε_9			$\alpha_5,\ \varepsilon_{10}$
$\{2,4\}$		$\varepsilon 11$		$\varepsilon_{12},\ \alpha_6$	
$\{3,4\}$		$\varepsilon 13$	$\alpha_7,\ \varepsilon_{14}$		
$\{1,3\}$		α_8	$\varepsilon 15$		$\varepsilon 16$
$\{2,3\}$		α_9	$\varepsilon 17$	$\varepsilon 18$	
$\{1,2\}$	$\alpha 10$			$\varepsilon 19$	$\varepsilon 20$

Table 2: Elements of $J_4(T_5)$ categorized by images missing a single element

Cite this article as:
Lawal, Z., Ali, B., & Abubakar, U.I. (2025) A graph-theoretic characterisation of generating sets in finite full transformation semigroups. *FNAS Journal of Mathematical Modeling and Numerical Simulation*, *2*(4), 45-50. https://doi.org/10.63561/jmns.v2i4.1122

$Where$: $\alpha_1 = \begin{pmatrix} 15 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 25 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 & 35 & 4 \\ 1 & 2 & 4 & 5 \end{pmatrix}$,
$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 45 \\ 1 & 3 & 4 & 5 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 14 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix}, \alpha_6 = \begin{pmatrix} 14 & 2 & 3 & 5 \\ 1 & 3 & 4 & 5 \end{pmatrix}$,
$\alpha_7 = \begin{pmatrix} 1 & 2 & 34 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix}, \alpha_8 = \begin{pmatrix} 13 & 2 & 4 & 5 \\ 1 & 2 & 3 & 5 \end{pmatrix}, \alpha_9 = \begin{pmatrix} 1 & 23 & 4 & 5 \\ 1 & 2 & 3 & 5 \end{pmatrix}$,
$\alpha_{10} = \begin{pmatrix} 12 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

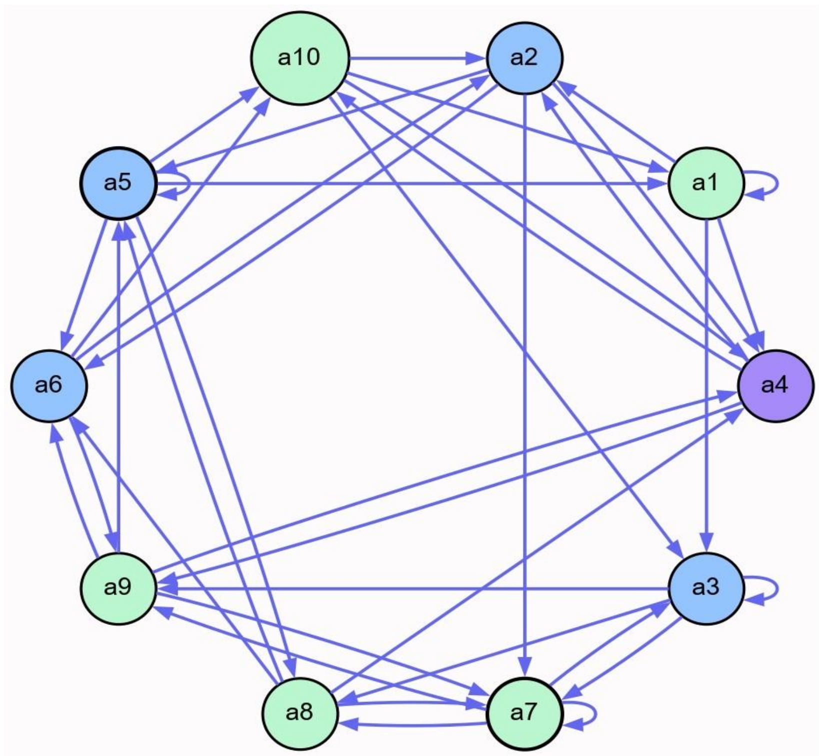

Figure 2: Directed graph of a Transformation Semigroup For n=5

It is observed that for n=4, six elements are enough to generate the semigroup. Also, for n=5, ten elements will generate. This coincides with Stirling number of the second kind will give the number of elements required to cover the graph in $Sing_n$

$$S(n, r) = S(n-1, r-1) + rs(n-1, r)\ S(n, 1) = S(n, n) = 1$$

Conclusion

In this study, we have established a graph-theoretic framework for analyzing generating sets in the semigroup of full transformations. By associating each subset, $A \subseteq D_{n-1}$ with a digraph Γ_A, we demonstrated that the algebraic property of generation is equivalent to the structural condition that A forms a cover of D_{n-1} and that Γ_A is strongly connected. This equivalence provides a powerful link between semigroup theory and digraph connectivity, thereby offering a unified approach for characterizing generating systems. Furthermore, it was shown that minimal generating sets correspond precisely to minimal strongly connected covers, and that every generating subset induces a connected and cyclic digraph. Illustrative examples for $n = 4$ and $n = 5$ revealed that six and ten elements, respectively, suffice to generate the semigroup—values that coincide with the Stirling numbers of the

Cite this article as:
Lawal, Z., Ali, B., & Abubakar, U.I. (2025) A graph-theoretic characterisation of generating sets in finite full transformation semigroups. *FNAS Journal of Mathematical Modeling and Numerical Simulation*, *2*(4), 45-50. https://doi.org/10.63561/jmns.v2i4.1122

second kind. This observation highlights a deep combinatorial connection between the structure of transformation semigroups and classical enumeration theory. The results obtained contribute to the broader understanding of how algebraic generation, combinatorial coverings, and graph connectivity interplay within finite full transformation semigroups. Future investigations may extend these ideas to other transformation classes or explore analogous graph-theoretic characterizations in partial and Order-Preserving transformation semigroups.

Competing Interests

The authors declare that they have no competing interests.

References

Howie, J. M. (1978). Idempotent generators in finite full transformation semigroups, *Proc. R. Soc. Edinburgh A*, 78,

Cameron, P. J., Castillo-Ramirez, A., Gadouleau, M., & Mitchell, J. D., (2017). Lengths of words in transformation semigroups generated by digraphs", *Journal of Algebraic Combinatorics*

East, J., Gadouleau, M., & Mitchell, J. D. (2019). Structural aspects of semigroups based on digraphs, *Algebraic Combinatorics*, 2019.

Jonuˇsas, J., & Troscheit, S. (2017). Random ubiquitous transformation semigroups", arXiv:1705.05709

Lallement, G. (1979). *Semigroups and Combinatorial Applications*, Wiley, 1979.

Gomes, G., & Howie, J. (1980). Idempotent-generated semigroups of transformations", *Proc. Edinburgh Math. Soc.*

Cite this article as:
Zaharadeen, L., Ali, B., & Abubakar, U.I. (2025) A graph-theoretic characterisation of generating sets in finite full transformation semigroups. *FNAS Journal of Mathematical Modeling and Numerical Simulation, 2*(4), 45-50. https://doi.org/10.63561/jmns.v2i4.1122