



A Fourth-Order Compact Finite Difference–Boole’s Method for Solving Linear Integro-Differential Equations

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Abstract

In this paper, the fourth order Compact Finite Difference – Boole’s Method was used to solve linear Integro-differential equations of first and second orders respectively. The discretized unknown function in each case formed a system of linear algebraic equations and was subsequently solved using MATLAB package. The approximate solutions arrived at from the combined method have been compared with exact and other existing solutions of the given numerical problems which show that the proposed method is very easy, powerful and efficient in determining approximate solutions to linear integro-differential equations.

Keywords: Absolute, Error, Boole’s rule, CFDM, LIDEs.

Introduction

In sciences and engineering, different physical systems are modeled using integro-differential equations. Usually, it is difficult to obtain analytical solution to Integro-differential equations as such it is necessary to derive a scheme that can provide an approximate solution that is reliable. Different methods have been derived for this purpose but some have their limitations ranging from unrealistic assumptions, linearization, low convergent to divergent results. Among these methods include Chebyshev wavelets method, Adomian decomposition method, wavelet-Galerkin method, CAS wavelets method, sine-cosine wavelets, homotopy perturbation method, differential transform method and new homotopy Analysis method. Linear Integro-Differential Equation (LIDE) is an important branch of modern mathematics and arises often in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics (Kurt & Sezer, 2008). A variational iteration method and trapezoidal rule by Saadati et al (2008) was used for solving LIDEs. Heris (2012) applied modified Laplace Adomian decomposition method. Bashir and Sirajo (2020), used finite difference Simpson’s approach on Fredholm Integro-differential equation and proved the error estimation of the method. In the work of Aruchunan and Sulaiman (2011), a numerical solution of first order linear Fredholm integro-differential equations was obtained using conjugate gradient method. A reliable algorithm with application was presented by Al-Towaiq and Kasasbeh (2017).

Compact finite Difference Methods to approximate solutions to such problems, especially in the context of the ordinary and partial differential equations has attracted interest.

Comparatively, there has been less progress made in determining high-order Compact Finite Difference Method in terms of integro-differential equations (IDE). Therefore, considerable works have been focusing on developing efficient high-order numerical schemes for approximating solutions of integro-differential equations. This work will examine the first order integro-differential equation:

$$f' = g(x) + \lambda \int_{a_0}^{a_1} K(x, t) f(t) dt$$

Noting that, a_0 , a_1 and λ are constants, f is unknown function while $g(x)$ and $K(x, t)$ (the Kernel) are known

functions .

And the second order equation of the form:

$$f'' = g(x) + \lambda \int_{a_0}^{a_1} K(x, t) f(t) dt$$

With dirichlet boundary conditions $f(a_0) = \alpha$ and $f(a_1) = \beta$.

Recently, Scientists have developed interest in applying higher-order numerical methods for solving ordinary differential equations (ODEs) and partial differential equations (PDEs). One scheme that has received attention is the application of the compact finite difference, which can achieve a high level of accuracy with a relatively few grid points. This scheme can result to more efficient and accurate solutions for ODEs and PDEs.

Several numerical solutions of the integro- differential equations have been studied by compact finite difference methods including Zhao and Coreless (2006) and Solimam et al (2012). A good number of researchers have developed numerical methods for integral and integro-differential equations recently as can be seen in Mirzaee and Hoseini (2017), Elahi Z. et al (2018), Sahu and Saha (2015), Chen and Zhang (2011), Taiwo and Jimoh (2014), Darania and Ebadian, (2007), Atkinson (1997), Zhao (2007), Bashir and Sirajo (2021), Aziz and Ain (2020) and Sameeh and Elsaid (2016).

This work will focus on the derivation of the combined fourth-order compact finite difference- Boole’s scheme for solving first order and second order integro-differential equations that are applied in different fields of human endeavour. The specific objectives are to efficiently handle the integral kernels and solve first and second order LIDEs using the derived Method and check how accuracy can be achieved.

The work is arranged thus: The first section considers the derivation of the scheme in detail. Two numerical experiments have been treated in the second section, the third section discusses the results obtained and conclusion is presented in the last section.

Derivation of the Scheme

To solve an integro-differential equation of the form

$$f' = g(x) + \lambda \int_{a_0}^{a_1} K(x, t) f(t) dt$$

With the initial condition $f(a_0) = \alpha$, we have to make use of $N \times N$ matrix.

From the standard Compact Finite Difference formula of the first derivative, if $i = 2, \dots, (N - 1)$, for interior points, since the boundary conditions are known, we need to adjust compact finite difference formula $i = 1$;

$$\frac{h}{3}(-17f'_1 - 14f'_2 + f'_3) = f_0 + 8f_1 - 9f_2$$

And when $i = N$, we can only use two points at the right hand side

$$\frac{h}{3}\left(\frac{1}{8}f'_{N-3} - \frac{5}{8}f'_{N-2} + \frac{19}{8}f'_{N-1} + \frac{9}{8}f'_N\right) = -f_{N-1} + f_N$$

In matrix form we have

$$M_1 F' = A_1 F + H_1$$

i.e

$$F' = M_1^{-1} A_1 F + M_1^{-1} H_1$$

Where

$$A_1 = \begin{pmatrix} 8 & -9 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 1 \end{pmatrix}_{(N)(N)}$$

$$H_1 = \begin{pmatrix} f_0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}_{(N)(N)}$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f_n \end{pmatrix}_{(N)(N)}$$

$$F' = \begin{pmatrix} f'_1 \\ f'_2 \\ \cdot \\ \cdot \\ \cdot \\ f'_{N-1} \\ f'_N \end{pmatrix}_{(N)(N)}$$

$$M_1 = \frac{h}{3} \begin{pmatrix} -17 & -4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ & & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 & 0 \\ & & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ & & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{19}{8} & \frac{9}{8} \\ & & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}_{(N)(N)}$$

The integral part in the equation will be treated with the quadrature method-the composite Boole’s rule represented as follows:

$$\int_a^b f(x)dx = \frac{2h}{45} \left[7(f(x_0) + f(x_n)) + 32 \left(\sum_{i \in 1,3,5,\dots,n-1} f(x_i) \right) + 12 \left(\sum_{i \in 2,6,10,\dots,n-2} f(x_i) \right) + 14 \left(\sum_{i \in 4,8,12,\dots,n-4} f(x_i) \right) \right]$$

Therefore, using the above rule

$$\int_a^b k_{ij}f(t)dt = \frac{2h}{45} \left[7(k_{ij}f(t_0) + k_{ij}f(t_n)) + 32 \left(\sum_{j \in 1,3,5,\dots,n-1} f(t_j) k_{ij} \right) + 12 \left(\sum_{j \in 2,6,10,\dots,n-2} k_{ij}f(t_j) \right) + 14 \left(\sum_{j \in 4,8,12,\dots,n-4} f(t_j) \right) \right]$$

Letting $f(t_j) = f_j$ then we have

$$\int_a^b k_{ij}f(t)dt = \frac{2h}{45} \left[7(k_{i0}f_0 + k_{in}f_n) + 32 \left(\sum_{j \in 1,3,5,\dots,n-1} k_{ij}f_j \right) + 12 \left(\sum_{j \in 2,6,10,\dots,n-2} k_{ij}f_j \right) + 14 \left(\sum_{j \in 4,8,12,\dots,n-4} k_{ij}f_j \right) \right]$$

Formulation with $(N \times N)$ Matrix

For $(N \times N)$, $f(0) = 0$, We have

$$A_1 M_1^{-1} F = g(x) + \frac{2h\lambda}{45} [7(k_{i0}f_0 + k_{in}f_n) + 32(\sum_{j \in 1,3,5,\dots,n-1} k_{ij}f_j) + 12(\sum_{j \in 2,6,10,\dots,n-2} k_{ij}f_j) + 14(\sum_{j \in 4,8,12,\dots,n-4} k_{ij}f_j)]$$

$$\text{Let } \gamma = \frac{2h\lambda}{45}$$

Expressing the right hand side in matrix form then for fourth order we have

$$\begin{aligned} A_1 M_1^{-1} F &= g(x) + \gamma B F \\ A_1 M_1^{-1} F - \gamma B F &= g(x) \\ F &= (A_1 M_1^{-1} - \gamma B)^{-1} * g(x) \end{aligned}$$

To solve an integro-differential equation of the form

$$f'' = g(x) + \lambda \int_{a_0}^{a_1} K(x, t) f(t) dt$$

With the boundary conditions mentioned above, it will be more convenient to make use of $(N - 1) \times (N - 1)$ matrix.

Formulation with $(N - 1)(N - 1)$ Matrix

Given the range $a_0 \leq x \leq a_1$ with N subintervals of space step size $h = \frac{a_1 - a_0}{N}$. Let x_i be the points of subdivision, noting that $x_i = a_0 + ih$ ($0 \leq i \leq N$) so that $x_0 = a_0$ and $x_N = a_1$. In this case, f_i is considered the approximate solution at x_i .

The second-order derivative is evaluated by the compact difference formula. The derivative of f is obtained by solving a diagonal dominance matrix system. For the second derivatives of $f(x)$ we have:

$$10f_i'' + (f_{i-1}'' + f_{i+1}'') = \frac{3}{h^2} (f_{i-1} - 2f_i + f_{i+1}) \quad (i = 1, 2, 3, \dots, N - 1)$$

For $i = 1$, we use

$$14f_1'' - 5f_2'' + 4f_3'' - f_4'' = \frac{3}{h^2} (f_0 - 2f_1 + f_2)$$

And when $i = N - 1$, the formula is

$$-f_{N-4}'' + 4f_{N-3}'' - 5f_{N-2}'' + 14f_{N-1}'' = \frac{3}{h^2} (f_{N-2} - 2f_{N-1} + f_N)$$

Both of the formulae are (h^4) . The matrix form is

$$A_2 F = M_2 F'' - H_2$$

Where

$$\begin{aligned} A_2 &= \frac{12}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 \end{pmatrix}_{(N-1)(N-1)} \\ H_2 &= \begin{pmatrix} u_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ u_n \end{pmatrix}_{(N-1)(N-1)} \\ M_2 &= \begin{pmatrix} 14 & -5 & 4 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ & 1 & 10 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 10 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 4 & -5 & 14 \end{pmatrix}_{(N-1)(N-1)} \end{aligned}$$

For $(N - 1)(N - 1)$, we have

$$(A_2 F + H_2) M_2^{-1} = g(x) + \frac{2h\lambda}{45} [7(k_{i0}f_0 + k_{in}f(t_n)) + 32(\sum_{j \in 1,3,5,\dots,n-1} k_{ij}f_j) + 12(\sum_{j \in 2,6,10,\dots,n-2} k_{ij}f_j) + 14(\sum_{j \in 4,8,12,\dots,n-4} k_{ij}f_j)]$$

$$\text{Let } \gamma = \frac{2h\lambda}{45}$$

Expressing the right hand side in matrix form then for fourth order using second derivative we have

$$\begin{aligned} A_2 M_2^{-1} F + M_2^{-1} H_2 &= g(x) + \gamma B F \\ A_2 M_2^{-1} F + M_2^{-1} H_2 - \gamma B F &= g(x) \\ F &= (A_2 M_2^{-1} - \gamma B)^{-1} * (g(x) - M_2^{-1} H_2) \end{aligned}$$

Note that when $\lambda = 0$,

$$f'' = g(x)$$

It implies that

$$\begin{aligned} A_2 M_2^{-1} F + M_2^{-1} F H_2 &= g(x) \\ A_2 M_2^{-1} F + M_2^{-1} H_2 &= g(x) \\ F &= (A_2 M_2^{-1})^{-1} * (g(x) - M_2^{-1} H_2) \end{aligned}$$

Numerical Experiments

To prove the validity of the proposed method, two examples have been considered, one on the first order and the other on the second order integro differential equations. The results obtained are analyzed using the error formula. The proposed method has been compared with the methods in [4] and [15] to show its higher accuracy.

Example 1 $f'(x) - \int_0^1 x f(t) dt = x e^x + e^x - x$, $f(0) = 0$
 $f(x) = x e^x$

Table 1: The Approximate Solutions and the Exact Solution

x	Exact	FDSM(Garba B. D., & Bichi, S. L. (2020)	CFDM(Proposed method)
0.1	0.11052	0.10884	0.11053
0.2	0.24428	0.24302	0.24447
0.3	0.40496	0.40174	0.40544
0.4	0.59673	0.59364	0.59762
0.5	0.82436	0.81899	0.82577
0.6	1.09327	1.08768	1.09532
0.7	1.40963	1.40138	1.41242
0.8	1.78043	1.77155	1.78411
0.9	2.21364	2.20166	2.21830
1.0	2.71828	2.70517	2.72404

Table 2: The errors of Differential Transform Method, FDSM and the proposed Method.

x	FDSM Error (Garba, B. D., & Bichi, S. L., 2020).	CFDM Error (Proposed method)	DTM Error (Daranian, P., & Ebadian, A., 2007)
0.1	1.6806×10^{-3}	1.63674×10^{-5}	1.00118×10^{-2}
0.2	1.2580×10^{-3}	1.94219×10^{-4}	2.78651×10^{-2}
0.3	3.2146×10^{-3}	4.80291×10^{-4}	5.08731×10^{-2}
0.4	3.0893×10^{-3}	8.88369×10^{-4}	7.55356×10^{-2}
0.5	5.3670×10^{-3}	1.40765×10^{-3}	9.71889×10^{-2}
0.6	5.5896×10^{-3}	2.04875×10^{-3}	1.09552×10^{-1}
0.7	8.2456×10^{-3}	2.80021×10^{-3}	1.041332×10^{-1}
0.8	8.8806×10^{-3}	3.67342×10^{-3}	6.94513×10^{-2}
0.9	1.1987×10^{-2}	4.65693×10^{-3}	1.00034×10^{-2}
1.0	1.3116×10^{-2}	5.76220×10^{-3}	1.55148×10^{-1}

Example 2 . $f''(x) = (x - 2) + 60 \int_0^1 (x - t)f(t)dt \quad 0 \leq x \leq 1.$
 $f(0) = 0 \quad f(1) = 0 \quad f(x) = x(x - 1)^2$

Table 3: The Exact and Approximation solutions of Example 2.

x	Exact u(x)	CFDM	Error
0.1	0.0810000000000000	0.081043223052295	4.322305×10^{-5}
0.2	0.1280000000000000	0.128068303094984	6.830309×10^{-5}
0.3	0.1470000000000000	0.147078441835646	7.844184×10^{-5}
0.4	0.1440000000000000	0.144076840981857	7.684098×10^{-5}
0.5	0.1250000000000000	0.125066702241195	6.670224×10^{-5}

0.6	0.0960000000000000	0.096051227321238	5.122732×10^{-5}
0.7	0.0630000000000000	0.063033617929562	3.361793×10^{-5}
0.8	0.0320000000000000	0.032017075773746	1.707577×10^{-5}
0.9	0.0090000000000000	0.009004802561366	4.802561×10^{-6}

Discussion of Results

Table 1 above shows the Exact , CFDM and FDSM solutions of Example 1.

In table 2, the absolute errors obtained from the proposed method, FDSM and DTM are compared and it is observed that the CFDM has a better accuracy than any of the existing methods mentioned.

In the table 3 above, it is also observed that the approximate solution obtained by our method is in close agreement with that of the exact solution of the problem in Example 2. The absolute error obtained indicated that our method can give good approximation to linear integro-differential equations.

Conclusion

The fourth order Compact Finite Difference Scheme has been combined with the Boole’s rule to solve linear Integro-differential equations. Two numerical experiments were considered to illustrate the procedure. It was observed that the approximate solutions obtained were in close agreement with those of the exact solutions which can be seen from the tables. MATLAB package was used to compute our exact and approximate solutions. The accuracy of the method was measured using the errors of the first order with FDSM in Garba and Bichi (2020). and the DTM in Darania and Ebadian (2007). The results obtained show that the method is highly accurate, remarkably effective and is very easy. It may also be concluded that the Method is very powerful and efficient in finding the analytical solutions of linear Integro-differential equations of first and second orders.

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