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Exact Solutions of Nonlinear Convolution-Type Differential Equations Using the Modified-Laplace Variational Iteration Method

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Abstract
This paper presents a novel application of the Modified-Laplace Variational Iteration Scheme (MLVIS) for solving nonlinear convolution differential equations. Convolution differential equations are integral to various fields, including engineering, physics, and applied mathematics, due to their ability to model systems with memory and hereditary properties. Traditional methods often struggle with the nonlinearity and complexity inherent in these equations. The approach combines the robustness of the Laplace transform with the flexibility of variational iteration methods, resulting in a powerful tool for tackling these challenges. This study begins by deriving the modified Laplace variational iteration method, emphasizing its theoretical foundation and practical implementation. The derivation process involves transforming the original nonlinear convolution differential equation into the Laplace domain, simplifying the convolution terms into algebraic forms. This transformation facilitates the application of variational principles, which are used to construct approximate solutions iteratively. The modified method enhances accuracy and convergence compared to standard techniques. The effectiveness of the MLVIM is validated through illustrative examples, demonstrating its capability to produce exact solutions for a range of nonlinear convolution differential equations. The results highlight the method's efficiency, precision, and potential for broader applications. The findings suggest that the MLVIM is a robust and reliable approach for solving complex nonlinear convolution differential equations.

Keywords: Modified-Laplace Variational Iteration Method, Nonlinear Convolution Differential Equations, Exact Solutions, Laplace Transform, Mathematical modelling.

Introduction

Nonlinear convolution-type differential equations are crucial in various scientific and engineering fields, modelling systems with memory effects and hereditary properties. These equations often arise in viscoelasticity, signal processing, and control theory, where the convolution terms represent the influence of past states on current dynamics. However, their nonlinear nature and complexity present significant challenges for conventional analytical and numerical methods, necessitating innovative approaches that can effectively and accurately address these challenges (Tarig & Ishag, 2013). Over the years, different numerical methods, such as the Variational Iteration Method (VIM) (Prakash et al., 2018; Michael et al., 2018; Vikramjeet et al., 2017; Yisa, 2018), and the Homotopy Perturbation Method (Amina et al., 2016; Liu, 2012; Yasir, 2011), have been employed to solve various differential equations. Studies have demonstrated that these methods are reliable and efficient for various scientific problems, including differential equations with initial or boundary conditions. Recently, there has been increasing interest in combining classical analytical techniques with iterative methods to solve complex differential equations. Such approaches are the combination of the Sumudu Transform with the Variational Iteration Method (STVIM), and the Laplace Transform with the Variational Iteration Method (LTVIM), which has proven particularly effective. The Sumudu transform simplifies the handling of convolution integrals by converting them into algebraic forms, reducing the problem's complexity (Vilu et al., 2019; Yindoula et al., 2020).

Meanwhile, the Variational Iteration Method (VIM), introduced by Ji-Huan He, provides a systematic method for constructing correction functionals that iteratively refine approximate solutions, enhancing accuracy and convergence (Abbasbandy & Elyas, 2009). This paper introduces a novel combination of the Modified-Laplace transform and the Variational Iteration Scheme (MLVIM) to solve nonlinear convolution-type differential equations. The key advantage of this combined approach is its ability to harness the strengths of both methods: the modified-Laplace transform's power in handling differential equations algebraically and the VIM's iterative

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decompose nonlinear terms in the equations and refinement to achieve accurate solutions. This method not only simplifies the solution process but also improves convergence to exact solutions for a wide range of nonlinear convolution-type differential equations.

Definitions

1. Laplace Transform Method (LTM):

Ruslan (2020), explains that Laplace transform of a function f (t), defined for all real numbers t ≥ 0, results in a function *F*(*s*), which is a unilateral transform given as

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt \ ,(R(s) > 0), \tag{1}$$

here, the function f (t) is assumed to be piece-wise continuous and of exponential order.

2. Modified-Laplace Transform Method

Faisal et al. (2020) defined the modified-Laplace transform of a function f (t) as

$$L(f(t)) = \int_0^\infty a^{-st} f(t)dt \ ,(R(s) > 0), \tag{2}$$

Theorems

1. If $f(t)$ is a piecewise continuous function of exponential order, where Equation (1) is modified -Laplace transform of $f(t)$ and $f^z(t)$ is the z-th order derivative of a function $f(t)$, then modified-Laplace transform of z-th order derivative is

$$L_a(f^z) = s^z(\log_e a)^z L_a f(t) - \sum_{k=1}^{z} s^{z-k}(\log_e a)^{z-k} f^{k-1}(0) \tag{3}$$

Where f^z is the z-th order derivative

Proof: Applying the principle of mathematical induction, when z=1 in Equation (3),

$$L_a(f') = s(\log_e a)L_a(f(t)) - f \ (0) \quad . \tag{4}$$

To achieve Equation (4),

$$L_a(f'(t)) = \int_0^\infty a^{-st}(f'(t))dt \tag{5}$$

Simplifying Equation (5), using the method of integration by parts, gives

$$L_a(f'(t)) = -f(0) + s\log_e a \int_0^\infty f(t)\, a^{-st} dt \, ; \tag{7}$$

Hence,

$$L_a(f'(t)) = s(\log_e a)L_a(f(t)) - f(0) \, . \tag{8}$$

And so z =1 is true.

Assuming z=m in Equation (3) is true, then

$$L_a(f^m(t)) = s^m(\log_e a)^m L_a(f(t)) - \sum_{k=1}^{m} s^{m-k}(\log_e a)^{m-k} f^{k-1}(0) \ , \tag{9}$$

is assumed to be true. It has to be shown that z=m+1 in Equation (3) is true whenever z=m is true. That is

$$L_a(f^{m+1}(t)) = s^{m+1}(\log_e a)^{m+1} L_a(f(t)) - \sum_{k=1}^{m+1} s^{m+1-k}(\log_e a)^{m+1-k} f^{k-1}(0), \tag{10}$$

is true.

to show Equation (10), let

$$L_a(f^{m+1}(t)) = \int_0^\infty a^{-st}(f^{(m+1)}(t))dt \tag{11}$$

Simplifying Equation (11) with method of integration by part, then

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$$L_a\left(f^{m+1}(t)\right) = -f^m(0) + s\log_e a\int_0^\infty a^{-st} f^m(t)dt \tag{12}$$

Since

$$L_a\left(f^m(t)\right) = \int_0^\infty a^{-st}\left(f^{(m)}(t)\right)dt \tag{13}$$

Substituting Equation (13) into Equation (12), gives

$$L_a\left(f^{m+1}(t)\right) = -f^m(0) + s\log_e a\ L_a\left(f^m(t)\right) \tag{14}$$

Substituting Equation (9) into Equation (14), gives

$$L_a\left(f^{m+1}(t)\right) = s\log_e a\left(s^m(\log_e a)^m L_a(f(t)) - \sum_{k=1}^{m} s^{m-k}(\log_e a)^{m-k} f^{k-1}(0)\right) - f^m(0), \tag{15}$$

expanding Equation (15), gives

$$L_a\left(f^{m+1}(t)\right) = s^{m+1}(\log_e a)^{m+1} L_a(f(t)) - \sum_{k=1}^{m+1} s^{m+1-k}(\log_e a)^{m+1-k} f^{k-1}(0)\ . \tag{16}$$

This concludes the proof. The core idea of Theorem 1 is to determine the modified-Laplace transform of the derivative of a function with order z. This derivative is then employed to develop a scheme for solving the nonlinear differential equation of classical order z.

2. If f (t) and h(t) are piecewise continuous functions of exponential order, where

$$L_a(f(t)) = f(s,a) \tag{17}$$

And

$$L_a(h(t)) = h(s,a) \tag{18}$$

then the modified-Laplace transform of the convolution of these two functions is expressed as

$$L_a(f(t)*h(t)) = f(s,a)h(s,a) \tag{19}$$

Proof:

convolution of two functions f (t) and h(t) is defined as

$$f(t)*h(t) = \int_0^\infty f(q)h(t-q)dq\ . \tag{20}$$

Taking the modified-Laplace transform of convolution f (t) and h(t), gives

$$L_a(f(t)*h(t)) = \int_0^\infty f(t)*h(t)a^{-st}dt. \tag{21}$$

Substituting Equation (21) into Equation (20), gives

$$L_a(f(t)*h(t)) = \int_0^\infty\left(\int_0^\infty f(q)h(t-q)dq\right)a^{-st}dt\ . \tag{22}$$

Simplifying Equation (22), gives

$$L_a(f(t)*h(t)) = \int_0^\infty \int_0^\infty f(q)h(t-q)a^{-st}dtdq \tag{23}$$

Let z= t – q, dz = dt and t = z + q, Equation (23) becomes

$$L_a(f(t)*h(t)) = \int_0^\infty \int_0^\infty f(q)h(z)a^{-s(z+q)}dzdq\ . \tag{24}$$

Simplifying Equation (24), gives

$$L_a(f(t)*h(t)) = \int_0^\infty f(q)a^{-sq}dq\int_0^\infty h(z)a^{-s(z)}dz. \tag{25}$$

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Where

$$f(s,a)=\int_0^\infty f(q)a^{-sq}dq. \tag{26}$$

And

$$h(s,a)=\int_0^\infty h(z)a^{-sz}dz. \tag{27}$$

Substituting Equation (26) and Equation (27) into Equation (25), gives

$$L_a\left(f(t)*h(t)\right)=f(s,a)h(s,a). \tag{28}$$

This ends the proof. The significance of Theorem 2 lies in determining the modified-Laplace transform of the convolution of two functions. This result will be utilized to develop a method for solving nonlinear convolution-type differential equations.

The Scheme of MLVIM:

Consider the nonlinear differential equation in an operator form as

$$Ry(t)+Uy(t)+Ny(t)=f(t) \tag{29}$$

Where R represents the linear operator associated with the highest derivative, U denotes the remainder of the linear operator involving derivatives of lower order than R, N is a nonlinear operator and f (t) is nonhomogeneous term.

Applying the modified-Laplace transform of Equation (29), gives

$$L_a(Ry(t))=L_a(-Uy(t)-Ny(t)+f(t)) \tag{30}$$

Where

$$L_a(R(y(t)))=s^m(\log_e a)^m y(s\log_e a)-s^{(m-1)}(\log_e a)^{(m-1)}y(0)-\ldots-y^{(m-1)}(0) \tag{31}$$

Substituting Equation (31) into Equation (30), gives

$$\begin{aligned}(s\log_e a)^m y(s\log_e a)-s^{(m-1)}(\log_e a)^{(m-1)}y(0)-\ldots-y^{(m-1)}(0)=\\ L_a(-Uy(t)-Ny(t)+f(t))\end{aligned}. \tag{32}$$

Isolating $y(s\log_e a)$ in Equation (32), gives

$$y(s\log a)=\frac{y(0)}{s\log a}+\ldots+\frac{y^{(m-1)}(0)}{s^m(\log_e a)^m}+\left(\frac{-1}{s^m(\log_e a)^m}\right)L_a(-Uy(t)-Ny(t)+f(t)). \tag{33}$$

Taking the inverse modified-Laplace transform of Equation (33), gives

$$y(t)=L_a^{-1}\left(\frac{y(0)}{s\log a}+\ldots+\frac{y^{(m-1)}(0)}{s^m(\log_e a)^m}+\left(\frac{-1}{s^m(\log_e a)^m}\right)L_a(-Uy(t)-Ny(t)+f(t))\right)$$

(34)

Where initial approximation is

$$y_0(t)=L_a^{-1}\left(\frac{y(0)}{s\log a}+\ldots+\frac{y^{(m-1)}(0)}{s^m(\log_e a)^m}\right) \tag{35}$$

Simplifying Equation (35), becomes

$$y_0(t)=y(0)+\ldots+\frac{y^{m-1}(0)t^{m-1}}{(m-1)!}. \tag{36}$$

Initial approximation is equation (35) by substituting initial condition given into RHS of Equation (36) and simplifying.

Using variational iteration formula to Equation (30), gives

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$$y_{n+1} = y_n + \int_0^t \lambda(r)\left(Ry_n(r) + Uy_n(r) + Ny_n(r) - f(r)\right)dr \ , \tag{37}$$

Taking the modified-Laplace transform of Equation (37), gives

$$L_a(y_{n+1}) = L_a(y_n + \int_0^t \lambda(r)\left(Ry_n(r) + Uy_n(r) + Ny_n(r) - f(r)\right)dr) \ , \tag{38}$$

simplifying Equation (38), gives

$$y_{n+1}(s\log a) = y_n(s\log a) + \int_0^t \lambda(s)\begin{pmatrix}(s\log_e a)^m y(s\log_e a) - s^{(m-1)}(\log_e a)^{(m-1)} y(0) - \ldots - y^{(m-1)}(0) \\ + L_a(Uy_n(r) + Ny_n(r) - g(r))\end{pmatrix} dr \tag{39}$$

regarding $L_a(U\tilde{y}_n(r) + N\tilde{y}_n(r))$ as restricted terms and differentiating Equation (39) with respect to $y_n(s\log_e a)$ and setting $\frac{dy_{n+1}(s\log_e a)}{dy_n(s\log_e a)} = 0$, gives

$$\left(\lambda(s)(s\log_e a)^m\right)_0^t + 1 = 0 \ . \tag{40}$$

Isolating λ (t) in Equation (40), gives

$$\lambda(t) = \frac{-1}{(s\log a)^m} \ . \tag{41}$$

Substituting Equation (41) into Equation (39), gives

$$y_{n+1}(s\log a) = y_n(s\log a) + \left(\frac{-1}{(s\log a)^m}\right)\left(L_a(Ry_n(t) + Uy_n(t) + Ny_n(t) - f(t))\right) , \tag{42}$$

The successive approximations are derived by applying the inverse modified-Laplace transform to Equation (42), resulting in the correction functional for Equation (29) as follows:

$$y_{n+1}(t) = y_n(t) + L_a^{-1}\left(\frac{-1}{(s\log a)^m}\left(L_a(Ry_n(t) + Uy_n(t) + Ny_n(t) - f(t))\right)\right) \ . \tag{43}$$

Illustrations:

1. Consider the nonlinear differential equation of convolution type given as (Tarig & Ishag 2013).

$$u'' - 2 + 2u' * u'' - u' * (u'')^2 = 0, \quad u(0) = u'(0) = 0, \tag{44}$$

Equation (44) has exact solution as

$$u(w) = w^2 \tag{45}$$

Convolution is a mathematical operation that combines two functions to produce a third function, illustrating how the shape of one function is altered by the other. The term "convolution" applies both to the resulting function and to the process of calculating it. It is represented by an integral that measures the overlap between the two functions as one is shifted over the other, effectively "blending" them together.

Initially, the MLVIM (Modified-Laplace Variational Iteration Method) will be used to solve this type of problem by applying the modified-Laplace transform to Equation (44) and incorporating the initial conditions.

$$u(s\log a) = \frac{1}{(s\log_e a)^2} L_a(2) + \frac{1}{s^2(\log_e a)^2} L_a\left(-2u' * u'' + u' * (u'')^2\right) \tag{46}$$

Taking the inverse modified-Laplace transform of Equation (46), becomes

$$u(w) = w^2 + L_a^{-1}\left(\frac{1}{(s\log_e a)^2} La\left(-2u' * u'' + u' * (u'')^2\right)\right) \tag{47}$$

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Using the MLVIM scheme of Equation (43) for Equation (44), then correction functional as

$$u_{n+1}(w) = u_n(w) + L_a^{-1}\left(\frac{-1}{(s\log_e a)^2} L_a\left(2 - 2u_n' * u_n'' + u_n' * (u_n'')^2\right)\right) \tag{48}$$

Starting with initial approximation $u_0(w) = w^2$ and using the iteration formula for Equation (48), gives

$$\left.\begin{aligned} u_1(w) &= w^2 \\ u_2(w) &= w^2 \\ u_3(w) &= w^2 \end{aligned}\right\} \tag{49}$$

Where $u_3(w)$ in Equation (49) is solution of Equation (44).

Table 1: Solutions of MLVIM and EXACT for Eqn. (44)

w	MLVIM (U)	Exact (U)
0.1	0.01000000	0.01000000
0.2	0.04000000	0.04000000
0.3	0.09000000	0.09000000
0.4	0.16000000	0.16000000
0.5	0.25000000	0.25000000
0.6	0.36000000	0.36000000
0.7	0.49000000	0.49000000
0.8	0.64000000	0.64000000
0.9	0.81000000	0.81000000
1.0	0.01000000	0.01000000

2. Consider the nonlinear differential equation of convolution type given as (Tarig & Ishag 2013).

$$u' + 2w - (u')^2 - u' * (u'')^2 = 0, \qquad u(0) = 1, \tag{50}$$

Equation (50) has exact solution

$$u(w) = 1 + w^2 \tag{51}$$

Taking the modified-Laplace transform of Equation (50), as well as substituting the initial conditions,

$$u(s\log a) = \frac{1}{s(\log_e a)} + \frac{1}{s(\log_e a)} L_a\left(-2w + (u')^2 - u' * (u'')^2\right) \tag{52}$$

Taking the inverse modified-Laplace transform of Equation (52), becomes

$$u(w) = 1 + L_a^{-1}\left(\frac{1}{s(\log_e a)} L_a\left(-2w + (u')^2 - u' * (u'')^2\right)\right) \tag{53}$$

Using the MLVIM scheme of Equation (43) for Equation (50), then correction functional as

$$u_{n+1}(w) = u_n(w) + L_a^{-1}\left(\frac{-1}{s(\log_e a)} L_a\left(u_n' - (u_n')^2 + 2w - u_n' * (u_n'')^2\right)\right) \tag{54}$$

Starting with initial approximation $y_0(w) = 1$ and using the iteration formula for Equation (54), gives

$$\left.\begin{aligned} u_1(t) &= 1 + w^2 \\ u_2(t) &= 1 + w^2 \\ u_3(t) &= 1 + w^2 \end{aligned}\right\} \tag{55}$$

Where $u_3(w)$ in Equation (55) is solution of Equation (50).

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Table 2: Solutions of MLVIM and EXACT for Eqn. (50)

w	MLVIM (U)	Exact (U)
0.1	1.01000000	1.01000000
0.2	1.04000000	1.04000000
0.3	1.09000000	1.09000000
0.4	1.16000000	1.16000000
0.5	1.25000000	1.25000000
0.6	1.36000000	1.36000000
0.7	1.49000000	1.49000000
0.8	1.64000000	1.64000000
0.9	1.81000000	1.81000000
1.0	1.01000000	1.01000000

Discussion

Theorems 1 and 2 establish the Modified-Laplace transform of an nth-order derivative and the convolution of two functions. These results were combined with the Variational Iteration Method (VIM) to develop a new modified scheme for solving nonlinear convolution-type differential equations. The proposed scheme was applied to several problems, with the results presented in Tables 1 and 2. The tables demonstrate that the scheme yielded exact solutions, indicating that the proposed method is a semi-analytical approach.

Conclusion

A new scheme for solving nonlinear convolution-type differential equations has been formulated by coupling modified-Laplace transform with VIM. Theorems associated with this scheme were also established. The modified-Laplace transform Lagrange multiplier for this method was also identified and effectively introduced into the derived scheme. Computational results showed that the method is effective and reliable. The method was used without linearization of functions, discretization of solution and unrealistic assumptions.

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