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Bernstein Perturbed Collocation Approach for Solving Fractional Integrodifferential Equations

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Abstract
Here, we propose the method of Bernstein perturbed collocation for the approximation of fractional integrodifferential equations. A trial solution of Bernstein polynomial was slightly perturbed by Chebyshev polynomial and put into the equation considered. The equation obtained was then collocated at equally spaced interior points of the interval, yielding a system of equations. These equations are subsequently solved using appropriate computational software, such as Maple 18. The methodology was demonstrated in some examples to illustrate the method's accuracy.

Keywords: Integrodifferential Equations, Perturbed Collocation Method, Bernstein Polynomials, Approach

Introduction

Fractional calculus constitutes a field of differentials and integrals of integers and non-integers (fractional) orders. With diverse applications spanning science, technology and engineering, there is an increased interest in utilizing fractional differentials and integro-differential equations to model real-world phenomena such as seismic activities and the regulation of electrical memory in sockets etc. However, solving fractional integro-differential equations (FIDEs) often comes with its challenges, as analytical solutions may not exist within the closed interval. Thus, resorting to approximate methods becomes necessary. A lot of techniques have been developed for this purpose, including the Adomian Decomposition Method (ADM), Standard. Least Squares Method (SLSM), Homotopy Analysis Transform Method (HATM), Collocation Method (CM), Homotopy Perturbation Method (HPM), Finite Difference Method (FDM), Finite Element Method (FEM), and Spectral Methods (SM), and others.

Many researchers have come up with innovative approaches to tackle problems involving fractional-order integro-differential equations. Ajileye et al. (2024) solved linear and nonlinear Fredholm integro-differential equations numerically. They used standard collocation points to convert their equation to a set of equations. The algebraic equations were then solved using the matrix inversion approach. The method was found to be e ffcient and accurate. Aduroja et al. (2023) examined the collocation approximation method for the solution of some classes of Volterra integro-differential equations with polynomial basis functions. The matrix approach was used to obtain the required algebraic equation and after some transformations, the authors were able to obtain a system of equations. To get the numerical approximation, they simply substituted their constants obtained into the trial solution and this yielded accurate solutions. Ajileye et al.(2023) considered the collocation approach for the solution of the Volterra integro-differential equation using the Chebyshev polynomial base function. They converted the equation to some linear system of equations and solved the matrix obtained to get the values of the constants in the equations. Results obtained at the end showed that the method is efficient. Ajileye et al. (2022) applied a collocation approach to solve Volterra-Fredholm Integro-differential equations. The authors transformed the equation into a set of equations by matrix method to be able to solve the system of equations. Alshbool et al. (2022) developed two techniques of Bernstein fractional polynomials, namely the fractional Bernstein operational matrix method and operational matrices of differentiation method.

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The two methods were used to solve fractional integro-differential equations (FIDEs). The schemes were introduced based on the idea of operational matrices generated using integration and operational matrices of differentiation respectively. By collocation, the authors applied the Riemann-Liouville and fractional derivative in Caputo's sense on Bernstein polynomials to obtain the approximate solutions of the proposed FIDEs. Also, the residual correction procedure for both methods were provided to estimate the absolute errors. The results were found to be good and converged to the exact solution. Adebisi et al., (2021) investigated the application of perturbation on the Galerkin method for the solution some classes of fractional integro-differential equations.

Uwaheren et al. (2022) worked on Akbari-Ganji's method to solve Volterra type of integro-differential difference equations. At the end, the approximate and the exact solutions were compared and the results showed that there was high convergence. Uwaheren et al. (2021) applied the Legendre Galerkin method for solving fractional integro-differential equations of Fredholm type. Using the governing equation of the problem the authors were able to minimise the errors of the approximate solution without applying any other method to linearize the non-linear part of the problem. Oyedepo et al. (2021) studied the modified homotopy perturbation technique on fractional integrodifferential difference equations and Uwaheren et al. (2020) discussed multi-order fractional differential equations of Lane Emden type. Other authors who worked on related differential and integrodifferential equations include Shaher (2006) whose focus was on the solution of multi-order fractional differential equations, presented a new algorithm for the solution of linear and non-linear multi-order fractional differential equations based on the Adomian decomposition method, Oyedepo et al. (2016) and Avipsita et al. (2017) both worked on the solution of Volterra-type fractional order integrodifferential equations using the Bernstein polynomial basis, providing a robust method for handling this class of equations. However, the former utilized the method of least squares combined with Bernstein bases to solve fractional integro-differential equations and achieved high accuracy in the numerical solutions, Snayip (2016) presented the Bernstein-collocation method for the solution of nonlinear Fredholm and Volterra integrodifferential equations. By collocating and the matrix operations, the problem considered was reduced to a set of equations. The approximate solutions were obtained by solving the linear system of equations. The study also used the Bernstein series to solve nonlinear Fredholm and Volterra integro differential equations. The results were good compared to the exact solutions.

In this work, we considered the normal integro-differential equations with integer-ordered derivatives and exact solutions. These equations were solved at some non-integer (fractional) values at the neighbourhood of the normal integer ordered derivatives, n for $\lceil\alpha\rceil^{-} \leq n \leq \lceil\alpha\rceil^{+}$ where $\lceil\alpha\rceil^{-}$ and $\lceil\alpha\rceil^{+}$ indicate some carefully chosen fractional values to the left and right of n; the integer ordered derivatives. Their solutions are compared to the exact solution which was given at n all with the aim of ascertaining which of the $\lceil\alpha\rceil^{-}$ and $\lceil\alpha\rceil^{+}$ will converge better.

The general equation representing the problem to be considered is given as:

$$u^{(n)}(x) = f(x) + \int_0^x k(x,z)u(z)dz, \quad 0 \leq x, z \leq 1 \quad , \tag{1}$$

which we rewrite as

$$D^{\alpha}u(x) = f(x) + \int_0^x k(x,z)u(z)dz, \quad 0 \leq x, z \leq 1 \quad , \tag{2}$$

with the following supplementary conditions:

$$u^{(i)}(0) = \delta_i, \qquad i = 0,1,2,\ldots,n-1, \qquad n-1 < \alpha \leq n, \qquad n \in \mathrm{N}, \tag{3}$$

where $D^{\alpha}u(x)$ denotes the Caputo fractional derivative of $u(x)$, $f(x)$, $k(x,z)$ are given smooth functions; δ_i are real constants; x and z are real variables varying in [0,1]; and $u(x)$ is the unknown function to be determined.

Definition of Relevant Terms

Beta function:

Beta function is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

Where $x,y \in \mathrm{R}$

The gamma function is defined as

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$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (5)$$

This integral converges when the real part of z is positive ($\mathrm{Re}(z) > 0$).

(6)

$$\Gamma(1 + z) = z\Gamma(z)$$

When z is a positive integer

$$\Gamma(z) = (z - 1)! \quad (7)$$

Riemann-Liouville fractional integral: Riemann-Liouville fractional integral is de ned as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \mathrm{Re}(\alpha) > 0, \quad \mathrm{Re}(x) > 0, \quad \mathrm{Re}(t) > 0 \quad (8)$$

J^α denotes the fractional integral of order α.

Riemann-Liouville fractional derivative: Fractional order derivative in Riemann-Liouville sense denoted D^α is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)dt \quad (9)$$

n is a positive integer with the property that $0 < \alpha < n$.

Caputo Fractional Derivative: Fractional order Derivative in Caputo sense is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt \quad (10)$$

with some properties as:

1. $J^\alpha J^\beta f = J^{\alpha+\beta} f, \quad \alpha, \beta > 0$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}$, $a > 0, \beta > -1, x > 0$
3. $\mathcal{D}^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$
4. $J^\alpha \mathcal{D}^\alpha f(x) = \mathcal{D}^\alpha J^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0)\frac{x^k}{k!}$, $\quad x > 0, n - 1 < \alpha \leq n$
5. $D^\alpha C = 0$, where C is the constant,

Chebyshev Polynomial:

The Chebyshev polynomials of the first kind and of degree k are defined on the interval [-1, 1] as:

$$T_k(x) = \cos(k \cos^{-1}(x)) \quad (11)$$

and the recurrence relation is given as:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1,2,3,\cdots \quad (12)$$

The first few terms of the Chebyshev polynomials of degree n on the interval $[-1,1]$ are given as:

$T_0(x) = 1$	$T_3(x) = 4x^3 - 3x$
$T_1(x) = x$	$T_4(x) = 8x^4 - 8x^2 + 1$
$T_2(x) = 2x^2 - 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$

and the recurrence formula for shifted Chebyshev on the closed form interval [0,1] is:

$$T^*_{n+1}(x) = 2(2x-1)T^*_n(x) - T^*_{n-1}(x), \quad n \geq 1 \quad (13)$$

with some few terms given as:

$T_0^*(x) = 1$	$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$
$T_1^*(x) = 2x - 1$	$T_4^*(x) = 128x^4 - 256x^3 + 100x^2 - 32x + 1$
$T_2^*(x) = 8x^2 - 8x + 1$	$T_4^*(x) = 128x^4 - 256x^3 + 100x^2 - 32x + 1$

Bernstein Polynomial: The $(n + 1)$ Bernstein basis polynomials of degree n is defined as:

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$$B_{i,n}(x) := \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n \tag{14}$$

where $\binom{n}{i}$ is a binomial coefficient.

So, for example, $b_{2,5}(x) = \binom{5}{2} x^2 (1-x)^3 = 10x^2(1-x)^3$.

The first few Bernstein basis polynomials for blending 1, 2, 3, or 4 values together are:

$b_{0,0}(x) = 1$ $b_{0,1}(x) = 1 - x$ $b_{1,1}(x) = x$ $b_{0,2}(x) = (1-x)^2$ $b_{1,2}(x) = 2x(1-x)$ $b_{2,2}(x) = x^2$ $b_{0,3}(x) = (1-x)^3$ $b_{1,3}(x) = 3x(1-x)^2$ $b_{2,3}(x) = 3x^2(1-x)$ $b_{3,3}(x) = x^3$

A combination of Bernstein polynomials:

$$u_n(x) := \sum_{i=0}^{n} a_i B_{i,n}(x) \tag{15}$$

equation (15) is an n^{th}-degree Bernstein polynomial.

Methodology

Our approach is based on approximating the unknown function $u(x)$ in equation (1) which is rewritten as in (2). To solve equation (2), we used a trial approximant of the form:

$$u_N(x) = \sum_{i=0}^{N} a_i B_{i,N}(x) \tag{16}$$

Equation (16) is slightly perturbed to get

$$u_N(x) = \sum_{i=0}^{N} a_i B_{i,N}(x) + G_n(x) \tag{17}$$

where

$$G_n(x) = \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x) \tag{18}$$

is called the perturbation term, $\lceil\alpha\rceil$ is the smallest integer which is bigger than α, which is The order of the fractional integro-differential equation. N is the degree of the approximation, $T^*(x)$ is the shifted Chebyshev polynomial basis function. $\tau_v (v = 1(1)n)$ are the free tau parameters to be determined and $a_i, i = 0,1,2,...N$ are the unknown constants also to be determined.

Substituting (17) into (2) gives

$$D^\alpha \left[\sum_{i=0}^{N} a_i B_{i,N}(x) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x) \right] = f(x) + \int_0^x k(x,t) \left[\sum_{i=0}^{N} a_i B_{i,N}(t) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(t) \right] dt \quad (19)$$

Applying J^α on both sides of equation (19) yields

$$J^\alpha \mathcal{D}^\alpha \left[\sum_{i=0}^{N} a_i B_{i,N}(x) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x) \right] = \sum_{k=0}^{n-1} u^k(0) \frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \int_0^x k(x,t) [\sum_{i=0}^{N} a_i B_{i,N}(t) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(t)] dt \quad (20)$$

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which yields

$$\sum_{i=0}^{N} a_i B_{i,N}(x) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x) = \sum_{k=0}^{n-1} u^k(0)\frac{x^k}{k!} + J^\alpha f(x) + J^\alpha \int_0^x k(x,t)\left[\sum_{i=0}^{N} a_i B_{i,N}(t) + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(t)\right] dt \quad (2$$

Equation (21) is further simplified and then collocated at equally spaced interior points, $t = t_v \text{ on } [a,b]; \quad t_v = a + \frac{(b-a)i}{N}; \quad i = 1, 2, ..., N$, to obtain a set of linear equations. The set of equations is solved using a computer package; Maple 18, to obtain the unknown constants. The obtained answers are then substituted back into the assumed approximate solution (16) to give the required approximate solution.

Numerical Examples

First Example:

Consider the following fractional integrodifferential equation

$$u''(x) = -x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)\,dt, \quad u(0) = 0, \quad u'(0) = 2 \qquad (22)$$

with the exact solution $u(x) = x + sin(x)$

We rewrite equation (22) as

$$\mathcal{D}^\alpha u(x) = -x - \frac{1}{3!}x^3 + \int_0^x (x-t)u(t)\,dt, \quad u(0) = 0, \quad u'(0) = 2 \qquad (23)$$

Taking a trial solution for N = 3,

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1 \cdot 3x(1-x)^2 + a_2 \cdot 3x^2(1-x) + a_3x^3 \qquad (24)$$

Eqn (24) is perturbed as

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1 \cdot 3x(1-x)^2 + a_2 \cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x) \qquad (25)$$

substituting (25) into (23) and

$$D^\alpha\left[a_0(1-x)^3 + a_1 \cdot 3x(1-x)^2 + a_2 \cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x)\right] = -x - \frac{1}{3!}x^3 +$$

$$\int_0^x (x-t)\left[a_0(1-t)^3 + a_1 \cdot 3t(1-t)^2 + a_2 \cdot 3t^2(1-t) + a_3t^3 + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(t)\right] dt \quad (26$$

Now, apply J^α, for $\alpha = \frac{5}{2}$, with the given initial conditions. After some further simplification, we have

$$\begin{aligned}
&- 0.003206405a_0x^{13/2} - 0.0191048324x^{9/2}\tau_2 - 0.0004275207x^{15/2}a_3 - 0.0085504145x^{13/2}\tau_2 \\
&+ 0.006412810a_1 x^{13/2} + 0.010420817a_0 x^{11/2} - 0.00694721x^{11/2}\tau_1 - 0.06252490x^{11/2}\tau_3 \\
&- 0.038209664a_0 x^{9/2} - 0.001282562x^{15/2}a_1 - 0.0032064054a_2 x^{13/2} + 0.027788847x^{11/2}\tau_2 \\
&+ 0.019104832x^{9/2}\tau_3 + 0.000427520x^{15/2}a_0 - 0.013680663x^{15/2}\tau_3 + 0.0012825621x^{15/2}a_2 \\
&- 0.0191048324x^{9/2}\tau_0 + 0.051302487x^{13/2}\tau_3 - 0.010420817a_1 x^{11/2} + 0.019104832x^{9/2}\tau_1 \\
&- 6a_1 x^2 + 3x^2a_0 + 8x^2\tau_2 - 48x^2\tau_3 - a_0 x^3 + 3a_1 x^3 + 32x^3\tau_3 - 3xa_0 + 18x\tau_3 - 8x\tau_2 - 3x^3a_2 + \\
&2\tau_1 x - \tau_1 + \tau_2 - \tau_3 + \tau_0 + 3xa_1 + 3x^2a_2 + a_0 + x^3a_3 = 2x - 0.08597174607x^{7/2} - 0.0034736059x^{11/2}
\end{aligned} \qquad (27)$$

Equation (27) is collocated at some equally spaced interior points in the interval [0, 1], which results in 8 sets of linear equations. We solve the equations and then substitute the answers into the assumed approximate equation (24), we get

$$u_3(x) = -0.1168506545x^3 + 0.0469419635x^2 + 1.989348036x + 0.00073065454$$

Following the same process prescribed in the methodology, the approximate solution is obtained for $\alpha = \frac{3}{2}$ as,

$$u_3(x) = -0.07450x^3 - 0.24850x^2 + 2.04450x - 0.00327$$

Second Example:

Consider the following fractional integrodifferential equation

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$$u''(x) = x + \int_0^x (x-t)u(t)\,dt, \quad u(0)=0, \quad u'(0)=1 \tag{28}$$

with the exact solution $u(x) = \sinh(x)$. we rewrite equation (28) as

$$D^\alpha u(x) = x + \int_0^x (x-t)u(t)\,dt, \quad u(0)=0, \quad u'(0)=1 \tag{29}$$

Taking an approximate solution for N = 3, we have

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1\cdot 3x(1-x)^2 + a_2\cdot 3x^2(1-x) + a_3x^3 \tag{30}$$

Equation (30) is perturbed as

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1\cdot 3x(1-x)^2 + a_2\cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{[\alpha]} \tau_v T_v^*(x) \tag{31}$$

substituting (31) into (29), yields

$$D^\alpha\left[a_0(1-x)^3 + a_1\cdot 3x(1-x)^2 + a_2\cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{[\alpha]} \tau_v T_v^*(x)\right] = x + \int_0^x (x-t)\left[a_0(1-t)^3 + a_1\cdot 3t(1-t)^2 + a_2\cdot 3t^2(1-t) + a_3t^3 + \sum_{v=0}^{[\alpha]} \tau_v T_v^*(t)\right]dt \tag{32}$$

Now, apply J^α, for $\alpha = \frac{5}{2}$, with the given initial conditions. After some further simplification, we have

$$\begin{aligned}
&0.0191048x^{9/2}\tau_3 - 0.0032064a_0x^{13/2} - 0.0004275207x^{15/2}a_3 + 0.006412811a_1x^{13/2} \\
&- 0.019104832x^{9/2}\tau_0 - 0.062525x^{11/2}\tau_3 - 0.01368066x^{15/2}\tau_3 - 6a_1x^2 + 0.0104208a_0x^{11/2} \\
&-0.038209664a_0x^{9/2} + 0.001283x^{15/2}a_2 - 0.010421a_1x^{11/2} - 0.008551x^{13/2}\tau_2 + 0.0191048x^{9/2}\tau_1 - \\
&0.00694721x^{11/2}\tau_1 + 0.027788847x^{11/2}\tau_2 - 0.003206405a_2x^{13/2} + 0.000427521x^{15/2}a_0 - 0.019104832x^{9/2}\tau_2 \\
&+0.05130249x^{13/2}\tau_3 - 0.00128256217x^{15/2}a_1 - 3xa_0 + 18x\tau_3 - 8x\tau_2 + 3x^2a_0 \\
&+8x^2\tau_2 - 48x^2\tau_3 - a_0x^3 + 3a_1x^3 + 32x^3\tau_3 + 3xa_1 + 3x^2a_2 - 3x^3a_2 + 2\tau_1x + \tau_0 - \tau_1 + \tau_2 - \tau_3 + x^3a_3 + a_0 \\
&= x + 0.08597174607x^{7/2}
\end{aligned} \tag{33}$$

Equation (33) is collocated at some equally spaced interior points in the interval [0, 1], which results in 8 set of linear equations. We solve the equations and then substitute the answers into the assumed approximate equation (30), we get

$$u_3(x) = 0.090853864x^3 - 0.028921592x^2 + 1.005981592x - 0.000373864$$

Following the same process prescribed in the methodology, the approximate solution is obtained for $\alpha = \frac{3}{2}$ as,

$$u_3(x) = 0.1928x^3 + 0.1385x^2 + 0.9897x + 0.0004$$

Third Example:

Consider the following fractional integrodifferential equation

$$u'''(x) = 1 + x - 2x^2 + \int_0^x (x-t)u(t)\,dt, \quad u(0)=5, u'(0)=u''(0)=1 \tag{34}$$

with the exact solution $u(x) = 4 + e^x$.

We rewrite equation (34) as

$$D^\alpha u(x) = 1 + x - 2x^2 + \int_0^x (x-t)u(t)\,dt, \quad u(0)=5, u'(0)=u''(0)=1 \tag{35}$$

Taking an approximate solution for N = 3,

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1\cdot 3x(1-x)^2 + a_2\cdot 3x^2(1-x) + a_3x^3 \tag{36}$$

Equation (36) is perturbed as

$$u_3(x) = \sum_{i=0}^{3} a_i B_{i,3}(x) = a_0(1-x)^3 + a_1\cdot 3x(1-x)^2 + a_2\cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{[\alpha]} \tau_v T_v^*(x) \tag{37}$$

substituting (37) into (35), yields

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$$D^{\alpha}[a_0(1-x)^3 + a_1 \cdot 3x(1-x)^2 + a_2 \cdot 3x^2(1-x) + a_3x^3 + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(x)] = 1 + x - 2x^2 +$$
$$\int_0^x (x-t)[a_0(1-t)^3 + a_1 \cdot 3t(1-t)^2 \quad + a_2 \cdot 3t^2(1-t) + a_3t^3 + \sum_{v=0}^{\lceil\alpha\rceil} \tau_v T_v^*(t)]\, dt \quad (38)$$

Now, apply J^{α}, for $\alpha = \frac{5}{2}$, with the given initial conditions. After some further simplification, we have

$$0.051302487x^{13/2}\tau_3 - 0.010420818a_1x^{11/2} - 0.0032064054a_0x^{13/2} - 0.019104832x^{9/2}\tau_0$$
$$+ 0.027788847x^{11/2}\tau_2 - 0.038209665a_0x^{9/2} + 0.001282562x^{15/2}a_2 - 0.019104832x^{9/2}\tau_2 +$$
$$0.0104208177a_0x^{11/2} - 0.00128256218x^{15/2}a_1 - 0.00855041453x^{13/2}\tau_2 + 0.019104832x^{9/2}\tau_3$$
$$-0.006947211x^{11/2}\tau_1 - 0.01368066x^{15/2}\tau_3 + 0.000427521x^{15/2}a_0 - 0.06254x^{11/2}\tau_3 - 0.000428x^{15/2}a_3$$
$$+0.01910x^{9/2}\tau_1 - 0.003206a_2x^{13/2} + 0.00641281a_1x^{13/2} - 6a_1x^2 - 3xa_0 + 18.0x\tau_3 - 8x\tau_2 + 3x^2a_0 +$$
$$8x^2\tau_2 - 48x^2\tau_3 - a_0x^3 + 3a_1x^3 + 32x^3\tau_3 + 3xa_1 + 3x^2a_2 - 3x^3a_2 + 2\tau_1x - \tau_1 + \tau_2 - \tau_3 + \tau_0$$
$$+ x^3a_3 + a_0 = 5 + x + 0.5x^2 + 0.30090111x^{5/2} + 0.085971746x^{7/2} - 0.0764193298x^{9/2} \quad (39)$$

Equation (39) is collocated at some equally spaced interior points in the interval [0, 1], which results in 8 sets of linear equations. We solve the equations and then substitute the answers into the assumed approximate equation (36), we get

$$u_3(x) = 0.233471421x^3 + 0.654975737x^2 + 0.981914263x + 5.001058579$$

Following the same process prescribed in the methodology, the approximate solution is obtained for $\alpha = \frac{7}{2}$, we obtained

$$u_3(x) = 0.07442823972x^4 + 0.0365657x^3 + 0.4967459x^2 + 1.0002431x + 4.9999913$$

For Tables see the appendix

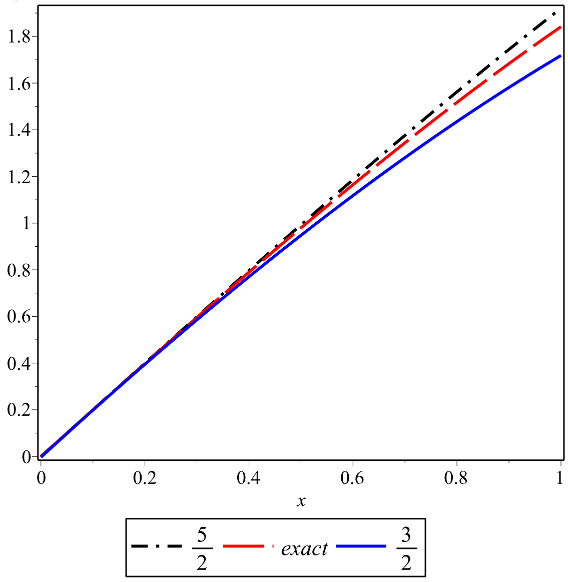

Figure 1: Error Representation of table in the first example

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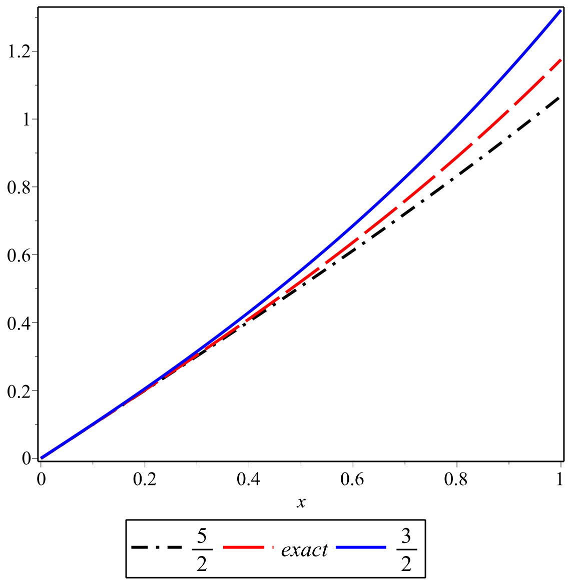

Figure 2: Error Representation of table in the second example

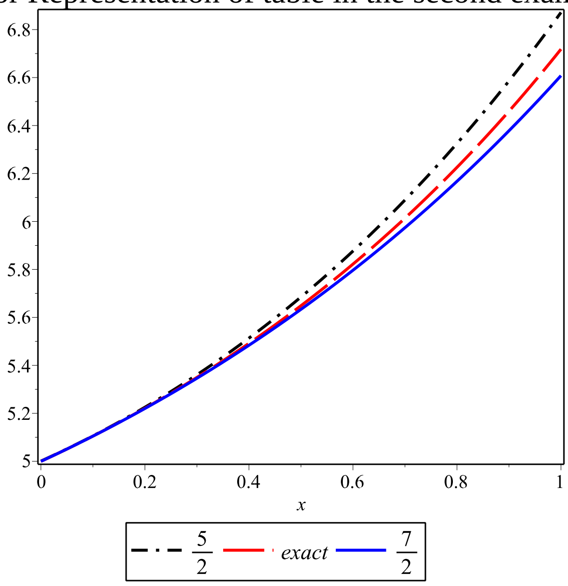

Figure 3: Error Representation of table in the third example

Discussion

In this study, the proposed method was used to solve Volterra fractional integro-differential equations successfully. Three examples were solved and the numerical results are presented in Tables 1, 2 and 3 and the graphs in Figures 1, 2 and 3. In Example 1, the results for $\alpha=\frac{5}{2}$were close to the exact solution and the result got closer for$\alpha=\frac{3}{2}$, which shows that the example,$\alpha=\frac{3}{2}$ performed better (converged) than$\alpha=\frac{3}{2}$. For example 2, the results $\alpha=\frac{5}{2}$converged closer to the exact solution than that of$\alpha=\frac{3}{2}$. In example 3, $\alpha=\frac{7}{2}$ performs better than$\alpha=\frac{5}{2}$. We note that the value of the exact solution as well as the source functions count as it was discovered that better results do not depend on whether $[\alpha]^-$ or $[\alpha]^+$ for it to converge better. The convergence occurred uniformly from the left or right within the interval [0,1]. Based on the numerical results, it is seen that the proposed method provides an acceptable estimation for the class of differential equations considered.

Conclusion

It must be noted that the problems studied in this article are normal integer differential equations solved at some non-integer (fractional) values at the neighbourhood of the normal integer ordered derivatives. So, it is cheerful to report that the perturbed collocation method using Bernstein basis functions offers a reliable approach for the solution of the class of fractional integrodifferential equations considered. The method is simple, accurate and less computational. Hence, we say that Bernstein basis functions with the collocation method is a good approximation tool. Incorporating perturbation in the collocation technique, effectively re fines and enhanced both the accuracy and convergence of solutions.

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Appendix

Table 1: Error table in first example

x	Exact Solution	Approximate Solution		Error	
		$\alpha = \frac{3}{2}$	$\alpha = \frac{5}{2}$	$\alpha = \frac{3}{2}$	$\alpha = \frac{5}{2}$
0.0	0.00000	-0.00327	0.00073	3.2700e-03	7.3065e-04
0.1	0.19983	0.19862	0.20002	1.2129e-03	1.8461e-04
0.2	0.39867	0.39509	0.39954	3.5753e-03	8.7380e-04
0.3	0.59552	0.58570	0.59860	9.8167e-03	3.0847e-03
0.4	0.78942	0.77000	0.79650	1.9416e-02	7.0838e-03
0.5	0.97943	0.94754	0.99253	3.1883e-02	1.3108e-02
0.6	1.16464	1.11788	1.18600	4.6764e-02	2.1356e-02
0.7	1.34422	1.28056	1.37620	6.3656e-02	3.1978e-02
0.8	1.51736	1.43515	1.56242	8.2210e-02	4.5068e-02
0.9	1.68333	1.58118	1.74398	1.0214e-01	6.0656e-02
1.0	1.84147	1.71823	1.92017	1.2324e-01	7.8699e-02

Table 2: Error table in second example

x	Exact Solution	Approximate Solution		Error	
		$\alpha = \frac{3}{2}$	$\alpha = \frac{5}{2}$	$\alpha = \frac{3}{2}$	$\alpha = \frac{5}{2}$
0.0	0.00000	0.00040	-0.00037	4.0000e-04	3.7386e-04
0.1	0.10017	0.10095	0.10003	7.8105e-04	1.4082e-04
0.2	0.20134	0.20542	0.20039	4.0864e-03	9.4358e-04
0.3	0.30452	0.31498	0.30127	1.0460e-02	3.2496e-03
0.4	0.41075	0.43078	0.40321	2.0027e-02	7.5464e-03
0.5	0.52110	0.55398	0.50674	3.2880e-02	1.4352e-02
0.6	0.63665	0.68572	0.61243	4.9071e-02	2.4226e-02
0.7	0.75858	0.82719	0.72080	6.8602e-02	3.7779e-02

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0.8	0.88811	0.97951	0.83242	9.1408e-02	5.5687e-02
0.9	1.02652	1.14387	0.94782	1.1735e-01	7.8701e-02
1.0	1.17520	1.32140	1.06754	1.4620e-01	1.0766e-01

Table 3: Error table in third example

x	Exact Solution	Approximate Solution		Error	
		$\alpha = \frac{5}{2}$	$\alpha = \frac{7}{2}$	$\alpha = \frac{5}{2}$	$\alpha = \frac{7}{2}$
0.0	5.00000	4.99999	5.00106	8.7000e-06	1.0586e-03
0.1	5.10517	5.10503	5.10603	1.4384e-04	8.6232e-04
0.2	5.22140	5.22032	5.22551	1.0814e-03	4.1055e-03
0.3	5.34986	5.34636	5.36088	3.4973e-03	1.1026e-02
0.4	5.49182	5.48381	5.51356	8.0112e-03	2.1738e-02
0.5	5.64872	5.63352	5.68494	1.5199e-02	3.6222e-02
0.6	5.82212	5.79651	5.87643	2.5609e-02	5.4309e-02
0.7	6.01375	5.97398	6.08942	3.9773e-02	7.5665e-02
0.8	6.22554	6.16731	6.32531	5.8230e-02	9.9771e-02
0.9	6.45960	6.37806	6.58551	8.1540e-02	1.2591e-01
1.0	6.71828	6.60797	6.87142	1.1031e-01	1.5314e-01

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