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Analysis of Convergence of Block Methods in Simulating Epidemic Diseases

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Abstract:
This research delves into a comprehensive examination of the application and convergence analysis of a newly developed block method for simulating epidemic models. The focal point of this study revolves around the derivation and implementation of a novel scheme, crafted through the utilization of power series polynomials, ensuring the fulfilment of essential properties. The formulation of the new scheme was rooted in the power series polynomial, a mathematical construct known for its versatility and precision. The rigorous validation process confirmed that the derived scheme satisfied the requisite properties, thereby establishing its theoretical soundness. The crux of the investigation lies in the practical application of this innovative scheme to simulate an epidemic model. Through meticulous simulations, the results yielded compelling evidence of the new method's superiority over existing approaches considered in this research. The comparative analysis demonstrated a notable enhancement in both accuracy and convergence speed, highlighting the efficacy of the newly proposed scheme in capturing and predicting the dynamics of epidemics. The observed advantages of the new scheme are particularly noteworthy, showcasing its potential to revolutionize the field of epidemiological modelling. By outperforming established methods, the new approach not only contributes to the theoretical underpinnings of epidemic modelling but also holds significant promise for practical applications, such as forecasting disease spread and optimizing intervention strategies.

Keywords: Accuracy, Basic Properties, Epidemical Models, New Scheme, Power Series.

Introduction

In numerical modelling of real-world issues across disciplines such as engineering, biological sciences, physical sciences, and electronics, initial value problems are frequently encountered (Shokri & Shokri, 2013). In epidemiological studies, the spread of infections over time or across populations can be effectively modelled using numerical methods, particularly through differential equations. The application of mathematical techniques to epidemic modelling is integral to applied sciences and various other fields. In these contexts, stochastic elements or "noises" are often incorporated into deterministic differential equation models to better capture the complex dynamics of disease transmission and progression (Kermack & Mckendrick, 1927). Epidemiological problems and studies are frequently expressed numerically and symbolically as equations, particularly differential equations, to provide meaningful frameworks for analysis, construction, and application. A foundational model in this domain is the Susceptible-Infective-Recovered (SIR) model, which was introduced by researchers Kermack and McKendrick (1927). Prominent researchers in the field of mathematical modelling translate the spread of transmissible viruses into differential equations. In these models, the population is divided into three categories: susceptible individuals (denoted by S), infective individuals (denoted by I), and removed or recovered individuals (denoted by R). Those in the recovered category are no longer at risk of becoming infected or spreading the infection. This could be due to recovery and subsequent immunity, vaccination, isolation from the population, or death. A disease that persists continuously within a population is termed endemic (Chasnov, 2009; Herbert, 1989).

Individuals in the recovered category of the SIR model are considered to have lifelong immunity. The SIR model is effectively described using ordinary differential equations (ODEs), representing a deterministic framework where identical initial conditions always produce the same outcomes. This model operates in continuous time rather than discrete intervals. According to the principles of response kinetics, interactions between infected and susceptible individuals occur at rates proportional to their respective numbers within the population (Herbert (1989); Misra, 2005). Modelling provides a straightforward way to illustrate how diseases spread over time. Many epidemic models focus on segmenting the population into a few distinct groups. The model is divided into three categories, which are as follows:

i. Susceptible individuals are those who harbour underlying conditions that compromise their immune system and render them resistant to treatment. Conversely, susceptibility also denotes the state wherein an individual is vulnerable to contracting a particular disease;
ii. Infected individuals are those who are ill and can readily pass on the infection to others.
iii. Recovered or Resistant (R): This category comprises individuals who have experienced illness for a duration and subsequently healed or recuperated. This includes illnesses where individuals gain enduring immunity.

This research concentrates on developing an algorithmic model employing a two-step approach to address a specific problem.

$$y' = y,\ y(0) = y_0,\ x \in [a, b] \tag{1}$$

Where $f: \Re \times \Re^m \to \Re^m, y, y_0 \in \Re^m, f$ is anticipated to meet the Lipchitz condition.

In the majority of instances, solving these initial value problems analytically proves impractical, necessitating the utilization of numerical methods. These methods are employed to derive an approximate solution for the initial value problem at hand (James et al., 2013).

Scholars have suggested various numerical methods to approximate initial value problems, spanning from discrete techniques (Lambert, 1973; Butcher, 2008; Fatunla, 1988), to methods employing prediction and correction (Kayode & Adeyeye, 2011; Adesanya et al., 2008; Awoyemi & Idowu, 2005) and subsequently, block techniques (Tumba et al., 2019; Sabo et al., 2019). Researchers have suggested various numerical techniques to solve equation (1.1), which can be categorized into single-step or multi-step methods. Multi-step methods come in the form of k-step or hybrid methods. The hybrid method, although challenging to formulate, has been noted to overcome the Dahlquist barrier condition by introducing off-step points. Despite its complexity, it provides superior approximation compared to the k-step method, particularly with smaller step lengths, (Odekunle et al., 2012). Reports suggest that the hybrid method offers improved stability conditions, particularly in cases where the problem exhibits stiffness or oscillations, (Omar et al., 2016; Skwame et al., 2012; Skwame et al., 2018).

The research is structured as follows: an introduction provides an overview, followed by the method's development in the second section. The analysis of the method is explored in the third section, while the fourth section applies the method mathematically to various physical problems. Finally, concluding remarks are drawn.

Materials and Methods

We utilize the power series polynomial as a fundamental function for developing the method, employing interpolation and collocation techniques.

The power series, serves as an approximation in the following format;

$$y(x) = h\sum_{i=0}^{u+v-1} \Omega_j \tau^i \tag{2}$$

be acknowledged, wherein u and v represent the points of interpolation and collocation, respectively. By applying differentiation to equation (2) once, we obtain

$$y'(x) = h\sum_{i=0}^{u+v-1} i\Omega_j \tau^{i-1} \tag{3}$$

Where $\Omega \in \Re$ for $i = 0\left(\frac{1}{7}\right)1$ and the function $y(x)$ possesses continuous differentiability. We shall endeavour to obtain the solution of equation (1.1) over the integration interval$[a, b]$, utilizing a constant step-size h which is specified by the relation $h = \chi_{n+1} - \chi_n,\ n = 0,\ 1,\ \cdots,\ N.$

Substituting equation (3) into (1) gives,

$$f(x, y) = h\sum_{i=0}^{u+v-1} i\Omega_j \tau^{i-1} \tag{4}$$

We interpolate equation (2) at point, $\tau_{n+u},\ u = \frac{1}{7}$ and collocate equation (4) at points

$\tau_{n+v},\ v=0,\ \frac{1}{7},\ \frac{2}{7},\ \frac{3}{7}\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ 1$ to give,

$M\Phi = N$ as

$$\begin{bmatrix} 1 & \tau_n & \tau_n^2 & \tau_n^3 & \tau_n^4 & \tau_n^5 & \tau_n^6 & \tau_n^7 & \tau_n^8 \\ 0 & 1 & 2\tau_n & 3\tau_n^2 & 4\tau_n^3 & 5\tau_n^4 & 6\tau_n^5 & 7\tau_n^6 & 8\tau_n^7 \\ 0 & 1 & 2\tau_{n+\frac{1}{7}} & 3\tau^2_{n+\frac{1}{7}} & 4\tau^3_{n+\frac{1}{7}} & 5\tau^4_{n+\frac{1}{7}} & 6\tau^5_{n+\frac{1}{7}} & 7\tau^6_{n+\frac{1}{7}} & 8\tau^7_{n+\frac{1}{7}} \\ 0 & 1 & 2\tau_{n+\frac{2}{7}} & 3\tau^2_{n+\frac{2}{7}} & 4\tau^3_{n+\frac{2}{7}} & 5\tau^4_{n+\frac{2}{7}} & 6\tau^5_{n+\frac{2}{7}} & 7\tau^6_{n+\frac{2}{7}} & 8\tau^7_{n+\frac{2}{7}} \\ 0 & 1 & 2\tau_{n+\frac{3}{7}} & 3\tau^2_{n+\frac{3}{7}} & 4\tau^3_{n+\frac{3}{7}} & 5\tau^4_{n+\frac{3}{7}} & 6\tau^5_{n+\frac{3}{7}} & 7\tau^6_{n+\frac{3}{7}} & 8\tau^7_{n+\frac{3}{7}} \\ 0 & 1 & 2\tau_{n+\frac{4}{7}} & 3\tau^2_{n+\frac{4}{7}} & 4\tau^3_{n+\frac{4}{7}} & 5\tau^4_{n+\frac{4}{7}} & 6\tau^5_{n+\frac{4}{7}} & 7\tau^6_{n+\frac{4}{7}} & 8\tau^7_{n+\frac{4}{7}} \\ 0 & 1 & 2\tau_{n+\frac{5}{7}} & 3\tau^2_{n+\frac{5}{7}} & 4\tau^3_{n+\frac{5}{7}} & 5\tau^4_{n+\frac{5}{7}} & 6\tau^5_{n+\frac{5}{7}} & 7\tau^6_{n+\frac{5}{7}} & 8\tau^7_{n+\frac{5}{7}} \\ 0 & 1 & 2\tau_{n+\frac{6}{7}} & 3\tau^2_{n+\frac{6}{7}} & 4\tau^3_{n+\frac{6}{7}} & 5\tau^4_{n+\frac{6}{7}} & 6\tau^5_{n+\frac{6}{7}} & 7\tau^6_{n+\frac{6}{7}} & 8\tau^7_{n+\frac{6}{7}} \\ 0 & 1 & 2\tau_{n+1} & 3\tau^2_{n+1} & 4\tau^3_{n+1} & 5\tau^4_{n+1} & 6\tau^5_{n+1} & 7\tau^6_{n+1} & 8\tau^7_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{7}} \\ f_n \\ f_{n+\frac{1}{7}} \\ f_{n+\frac{2}{7}} \\ f_{n+\frac{3}{7}} \\ f_{n+\frac{4}{7}} \\ f_{n+\frac{5}{7}} \\ f_{n+\frac{6}{7}} \\ f_{n+1} \end{bmatrix} \quad (5)$$

Solving (5), for $\Omega_i,\ i = 0\left(\frac{1}{7}\right)1$ and replacing back into (2) gives a linear block method as

$$y(x) = \alpha_{\frac{1}{7}}(x) y_{n+\frac{1}{7}} + h \begin{bmatrix} \beta_0(x) f_n + \beta_{\frac{1}{7}}(x) f_{n+\frac{1}{7}} + \beta_{\frac{2}{7}}(x) f_{n+\frac{2}{7}} + \beta_{\frac{3}{7}}(x) f_{n+\frac{3}{7}} + \\ \beta_{\frac{4}{7}}(x) f_{n+\frac{4}{7}} + \beta_{\frac{5}{7}}(x) f_{n+\frac{5}{7}} + \beta_{\frac{6}{7}}(x) f_{n+\frac{6}{7}} + \beta_1(x) f_{n+1} \end{bmatrix} \quad (6)$$

Evaluating (6) at non-interpolating points to gives

$$y_n = y_{n+\frac{1}{7}} - \frac{751}{17280} hf_n - \frac{139849}{846720} hf_{n+\frac{1}{7}} + \frac{4511}{31360} hf_{n+\frac{2}{7}} - \frac{123133}{846720} hf_{n+\frac{3}{7}} + \frac{88547}{846720} hf_{n+\frac{4}{7}} - \frac{1537}{31360} hf_{n+\frac{5}{7}} + \frac{11351}{846720} hf_{n+\frac{6}{7}} - \frac{275}{169344} hf_{n+1} \quad (7)$$

$$y_{n+\frac{2}{7}} = y_{n+\frac{1}{7}} - \frac{275}{169344} hf_n + \frac{5311}{94080} hf_{n+\frac{1}{7}} + \frac{11261}{94080} hf_{n+\frac{2}{7}} - \frac{44797}{846720} hf_{n+\frac{3}{7}} + \frac{2987}{94080} hf_{n+\frac{4}{7}} - \frac{1283}{94080} hf_{n+\frac{5}{7}} + \frac{2999}{846720} hf_{n+\frac{6}{7}} - \frac{13}{31360} hf_{n+1} \quad (8)$$

$$y_{n+\frac{3}{7}} = y_{n+\frac{1}{7}} - \frac{13}{6615} hf_n + \frac{1363}{26460} hf_{n+\frac{1}{7}} + \frac{46}{245} hf_{n+\frac{2}{7}} + \frac{1153}{26460} hf_{n+\frac{3}{7}} + \frac{52}{6615} hf_{n+\frac{4}{7}} - \frac{1}{196} hf_{n+\frac{5}{7}} + \frac{2}{1323} hf_{n+\frac{6}{7}} - \frac{1}{5292} hf_{n+1} \quad (9)$$

$$y_{n+\frac{4}{7}} = y_{n+\frac{1}{7}} - \frac{9}{6272} hf_n + \frac{337}{6272} hf_{n+\frac{1}{7}} + \frac{1107}{6272} hf_{n+\frac{2}{7}} + \frac{3897}{31360} hf_{n+\frac{3}{7}} + \frac{2777}{31360} hf_{n+\frac{4}{7}} - \frac{513}{31360} hf_{n+\frac{5}{7}} + \frac{117}{31360} hf_{n+\frac{6}{7}} - \frac{13}{31360} hf_{n+1} \quad (10)$$

$$y_{n+\frac{5}{7}} = y_{n+\frac{1}{7}} - \frac{8}{6615} hf_n + \frac{38}{735} hf_{n+\frac{1}{7}} + \frac{136}{735} hf_{n+\frac{2}{7}} + \frac{664}{6615} hf_{n+\frac{3}{7}} + \frac{136}{735} hf_{n+\frac{4}{7}} + \frac{38}{735} hf_{n+\frac{5}{7}} - \frac{8}{6615} hf_{n+\frac{6}{7}} \quad (11)$$

$$y_{n+\frac{6}{7}} = y_{n+\frac{1}{7}} - \frac{275}{169344} hf_n + \frac{9355}{169344} hf_{n+\frac{1}{7}} + \frac{1075}{6272} hf_{n+\frac{2}{7}} + \frac{22375}{169344} hf_{n+\frac{3}{7}} + \frac{22375}{169344} hf_{n+\frac{4}{7}} + \frac{1075}{6272} hf_{n+\frac{5}{7}} + \frac{9355}{169344} hf_{n+\frac{6}{7}} - \frac{275}{169344} hf_{n+1} \quad (12)$$

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$$y_{n+1} = y_{n+\frac{1}{7}} + \frac{41}{980}hf_{n+\frac{1}{7}} + \frac{54}{245}hf_{n+\frac{2}{7}} + \frac{27}{980}hf_{n+\frac{3}{7}} + \frac{68}{245}hf_{n+\frac{4}{7}} + \frac{27}{980}hf_{n+\frac{5}{7}} + \frac{54}{245}hf_{n+\frac{6}{7}} + \frac{41}{980}hf_{n+1} \quad (13)$$

Using equation (7) to make $y_{n+\frac{1}{7}}$ the subject of the formula, to get

$$y_{n+\frac{1}{7}} = y_n + \frac{751}{17280}hf_n + \frac{139849}{846720}hf_{n+\frac{1}{7}} - \frac{4511}{31360}hf_{n+\frac{2}{7}} + \frac{123133}{846720}hf_{n+\frac{3}{7}} - \frac{88547}{846720}hf_{n+\frac{4}{7}} + \frac{1537}{31360}hf_{n+\frac{5}{7}} - \frac{11351}{846720}hf_{n+\frac{6}{7}} + \frac{275}{169344}hf_{n+1} \quad (14)$$

Substitute (14) into (8) to (13) to gives the new scheme as

$$\left.\begin{aligned}
y_{n+\frac{1}{7}} &= y_n + \frac{751}{17280}hf_n + \frac{139849}{846720}hf_{n+\frac{1}{7}} - \frac{4511}{31360}hf_{n+\frac{2}{7}} + \frac{123133}{846720}hf_{n+\frac{3}{7}} - \frac{88547}{846720}hf_{n+\frac{4}{7}} + \frac{1537}{31360}hf_{n+\frac{5}{7}} + \frac{11351}{846720}hf_{n+\frac{6}{7}} + \frac{275}{169344}hf_{n+1}\\
y_{n+\frac{2}{7}} &= y_n + \frac{41}{980}hf_n + \frac{1466}{6615}hf_{n+\frac{1}{7}} - \frac{71}{2940}hf_{n+\frac{2}{7}} + \frac{68}{735}hf_{n+\frac{3}{7}} - \frac{1927}{26460}hf_{n+\frac{4}{7}} + \frac{26}{735}hf_{n+\frac{5}{7}} - \frac{29}{2940}hf_{n+\frac{6}{7}} + \frac{8}{6615}hf_{n+1}\\
y_{n+\frac{3}{7}} &= y_n + \frac{265}{6272}hf_n + \frac{1359}{6272}hf_{n+\frac{1}{7}} + \frac{1377}{31360}hf_{n+\frac{2}{7}} + \frac{5927}{31360}hf_{n+\frac{3}{7}} - \frac{3033}{31360}hf_{n+\frac{4}{7}} + \frac{1377}{31360}hf_{n+\frac{5}{7}} - \frac{373}{31360}hf_{n+\frac{6}{7}} + \frac{9}{6272}hf_{n+1}\\
y_{n+\frac{4}{7}} &= y_n + \frac{278}{6615}hf_n + \frac{1445}{6615}hf_{n+\frac{1}{7}} + \frac{8}{245}hf_{n+\frac{2}{7}} + \frac{1784}{6615}hf_{n+\frac{3}{7}} - \frac{106}{6615}hf_{n+\frac{4}{7}} + \frac{8}{245}hf_{n+\frac{5}{7}} - \frac{64}{6615}hf_{n+\frac{6}{7}} + \frac{8}{6615}hf_{n+1}\\
y_{n+\frac{5}{7}} &= y_n + \frac{265}{6272}hf_n + \frac{36725}{169344}hf_{n+\frac{1}{7}} + \frac{775}{18816}hf_{n+\frac{2}{7}} + \frac{4625}{18816}hf_{n+\frac{3}{7}} - \frac{13625}{169344}hf_{n+\frac{4}{7}} + \frac{1895}{18816}hf_{n+\frac{5}{7}} - \frac{275}{18816}hf_{n+\frac{6}{7}} + \frac{275}{169344}hf_{n+1}\\
y_{n+\frac{6}{7}} &= y_n + \frac{41}{980}hf_n + \frac{54}{245}hf_{n+\frac{1}{7}} + \frac{27}{980}hf_{n+\frac{2}{7}} + \frac{68}{245}hf_{n+\frac{3}{7}} + \frac{27}{980}hf_{n+\frac{4}{7}} + \frac{54}{245}hf_{n+\frac{5}{7}} + \frac{41}{980}hf_{n+\frac{6}{7}}\\
y_{n+1} &= y_n + \frac{751}{17280}hf_n + \frac{3577}{17280}hf_{n+\frac{1}{7}} - \frac{49}{640}hf_{n+\frac{2}{7}} + \frac{2989}{17280}hf_{n+\frac{3}{7}} + \frac{2989}{17280}hf_{n+\frac{4}{7}} + \frac{49}{640}hf_{n+\frac{5}{7}} + \frac{3577}{17280}hf_{n+\frac{6}{7}} + \frac{751}{17280}hf_{n+1}
\end{aligned}\right\} \quad (15)$$

Results

Basic Properties of the new Scheme

Order of the block method

To find the order and error constant of the new scheme, we first defined the linear difference operator L associated with equation (15) as

$$L[y(x);\ h] = M\Phi = N \quad (16)$$

Corollary (Omar & Adeyeye (2016))

Compared The Linear Operator (16) With The Truncation Error $C_{09}h^{09}y^{09}(x_n) + 0(h^{10})$.

Proof

The Linear Difference Operators (16) Is Compared With The New Scheme (15) As

$$\left.\begin{aligned}
l_{\frac{1}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{1}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{\frac{2}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{2}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{\frac{3}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{3}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{\frac{4}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{4}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{\frac{5}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{5}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{\frac{6}{7}}[y(x_n);h] &= y\left(x_n + \tfrac{6}{7}h\right) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)\\
l_{1}[y(x_n);h] &= y(x_n + h) - \left(\alpha_{\frac{1}{7}}\left(x_n + \tfrac{1}{7}h\right) + h\textstyle\sum_{j=0}^{1}(\beta_v(x)f_{n+v})\right)
\end{aligned}\right\} \quad (17)$$

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Corollary (Omar & Adeyeye, 2016)

The local truncation error of (15) is assume $y(x)$ to be sufficiently differentiable and expanding $y(x_n + qh)$ and $y(x_n + jh)$ about x_n using Taylor series to have

$$l_{\frac{1}{7}}[y(x_n); h] = (2.3186 \times 10^{-10}),\ l_{\frac{2}{7}}[y(x_n); h] = (1.8203 \times 10^{-10}),\ l_{\frac{3}{7}}[y(x_n); h] \tag{18}$$
$$= (2.0411 \times 10^{-10}), l_{\frac{4}{7}}[y(x_n); h] = (1.8706 \times 10^{-10}),\ l_{\frac{5}{7}}[y(x_n); h]$$
$$= (2.0914 \times 10^{-10}),\ l_{\frac{6}{7}}[y(x_n); h] = (1.5931 \times 10^{-10}), l_1[y(x_n); h]$$
$$= (3.9117 \times 10^{-10})$$

Proof

Expanding (17) using a Taylor series about x_n respectively and then collecting their like elements to the power of h gives

$$l_{\frac{1}{7}}[y(x_n); h] = (2.3186 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{2}{7}}[y(x_n); h]$$
$$= (1.8203 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{3}{7}}[y(x_n); h]$$
$$= (2.0411 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{4}{7}}[y(x_n); h]$$
$$= (1.8706 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{5}{7}}[y(x_n); h]$$
$$= (2.0914 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{6}{7}}[y(x_n); h]$$
$$= (1.5931 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_1[y(x_n); h]$$
$$= (3.9117 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10})$$

Hence, from the above results, the order of the new scheme is 9, and the error constants is

$$C_{10} = (2.3186 \times 10^{-10}\ 1.8203 \times 10^{-10}\ 2.0411 \times 10^{-10}\ 1.8706 \times 10^{-10}\ 2.0914 \times 10^{-10}\ 1.593$$
$$\times 10^{-10}\ 3.9117 \times 10^{-10})^T$$

Consistency (Omar & Adeyeye, 2016)

Now that the order $p \geq 7$, therefore, the new scheme (15) is consistent.

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Zero stability

Definition: The new scheme (15) is expected to be zero-stable, if the roots $z_s,\ s = 1,\ 2, \cdots,\ k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = det(zA - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 0$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0,\ \rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E, **(**Omar & Adeyeye, 2016).

Then our method,

$$\rho(z) = \left| z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^7(z-1)\rho(z) = z^7(z-1) = 0$$

$$l_{\frac{1}{7}}[y(x_n); h] = (2.3186 \times 10^{-10}),\ l_{\frac{2}{7}}[y(x_n); h] = (1.8203 \times 10^{-10}),\ l_{\frac{3}{7}}[y(x_n); h] = (2.0411 \times 10^{-10}), l_{\frac{4}{7}}[y(x_n); h] = (1.8706 \times 10^{-10}),\ l_{\frac{5}{7}}[y(x_n); h] = (2.0914 \times 10^{-10}),\ l_{\frac{6}{7}}[y(x_n); h] = (1.5931 \times 10^{-10}), l_1[y(x_n); h] = (3.9117 \times 10^{-10}) \quad (18)$$

Proof

Expanding (17) using a Taylor series about x_n respectively and then collecting their like elements to the power of h gives

$$l_{\frac{1}{7}}[y(x_n); h] = (2.3186 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{2}{7}}[y(x_n); h] = (1.8203 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{3}{7}}[y(x_n); h] = (2.0411 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{4}{7}}[y(x_n); h] = (1.8706 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{5}{7}}[y(x_n); h] = (2.0914 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_{\frac{6}{7}}[y(x_n); h] = (1.5931 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10}) l_1[y(x_n); h] = (3.9117 \times 10^{-10})h^9 y^{(9)}(x_n) + 0(h^{10})$$

Hence, from the above results, the order of the new scheme is 9, and the error constant is

$$C_{10} = (2.3186 \times 10^{-10}\ 1.8203 \times 10^{-10}\ 2.0411 \times 10^{-10}\ 1.8706 \times 10^{-10}\ 2.0914 \times 10^{-10}\ 1.593 \times 10^{-10}\ 3.9117 \times 10^{-10})^T$$

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Linear Stability

The concept of A-stability according to [12] is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \tag{19}$$

To yield

$$Y_m = \mu(z) Y_{m-1},\ z = \lambda h \tag{20}$$

Where $\mu(z)$ is the amplification matrix of the form

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^1\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^1\eta^{(1)}\right) \tag{21}$$

The matrix $\mu(z)$ has Eigen values $(0,\ 0,\ \cdots,\ \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of the method is given by

$$\zeta = -\frac{\begin{pmatrix}7612505820z^7 - 334\,745051466z^6 + 6620\,293799280z^5 - 102213\,085557541z^4 + 998319\,737297995z^3 \\ -7133554\,799947415z^2 + 29598119\,607974400z - 60247612\,422144000\end{pmatrix}}{\begin{pmatrix}9144576000z^7 - 331\,948108800z^6 + 7005\,050035200z^5 - 101102\,737075200z^4 + 1024619\,258880000z^3 \\ -7069872\,886272000z^2 + 30123806\,211072000z - 60247612\,422144000\end{pmatrix}} \tag{22}$$

Mathematical implementations

The new scheme was mathematically applied to a selection of problems, including the SIR model, the growth model, and a highly stiff differential equation. The results will be compared with those found in existing literature. The model is detailed below.

The following notations will be used in the tables and figures.

ER means exact result; CR means the computed result

ENS means absolute error in the new scheme;

EOA16 means absolute error in Omar and Adeyeye (2016);

ESOA13 means absolute error in Sunday et al. (2013);

ESA19 means absolute error in Sunday and Agbataobi (2019);

EOA16i means absolute error in the Two-Step Implicit Obrechkoff Method of Omar and Adeyeye (2016); EOA16ii means absolute error in the New Two-Step Obrechkoff-Type Block Method of Omar and Adeyeye (2016); EBYP15 means absolute error in Badmus et al. (2015).

Problem 1 (SIR Model):

The susceptible-infected-recovered (SIR) model represents an epidemiological framework that quantifies the aggregate of individuals afflicted by a communicable disease within a specified population over a designated temporal interval. Such models are predicated upon the premise that they encompass interrelated equations regarding the quantity of individuals who are vulnerable to the disease, denoted as S(t), the number of individuals currently experiencing the infection, represented as I(t), and the total count of individuals who have attained recovery, indicated as R(t). The model is interconnected as follows:

$$\frac{dS}{dt} = \mu - \beta SI - \mu S \tag{23}$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \tag{24}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{25}$$

For $\mu,\ \gamma\ and\ \beta$are positive parameters. $y$is given as,

$$y = S + I + R \tag{26}$$

Summing (23), (24) and (25) to get

$$y' = \mu(1 - y) \tag{27}$$

let $\mu = 0.5$with initial condition as $y(0) = -0.5$, we get,

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$$y = \frac{1}{2}(1-y),\ y(0) = \frac{1}{2},\ h = 0 \quad (28)$$

with exact solution:

$$y(t) = 1 - \frac{1}{2}e^{-t} \quad (29)$$

See: (Omar & Adeyeye, 2016; Sunday et al., 2013).

Problem 2 (Growth Model):
Consider a growth model, commonly referred to as a bacterial culture, which proliferates at a rate that is directly proportional to the total population present. After a duration of one hour, approximately 1000 bacterial strands are identified within the culture; subsequently, after four hours, 3000 strands of the bacteria are also observed. Determine the cumulative total of bacterial strands present in the culture at a specified time. t: $0 \le t \le 1$
We assumed $y(t)$ to be the number of bacteria strands in the culture at time$t$, the equation is modeled as

$$y' = 0.366y,\ y(0) = 694 \quad (30)$$

With exact solution as,

$$y(t) = 694e^{0.366t} \quad (31)$$

See: (Sunday & Agbataobi, 2019).

Problem 3

Consider the Highly stiff differential equation

$$y' = \lambda\eta,\ h = 0.1,\ t(0) = \lambda = 1,\ \eta = -1 \quad (32)$$

With the exact solution

$$y(t) = exp(\eta t) \quad (33)$$

Source: (Omar & Adeyeye, 2016; Badmus et al., 2015).

Results

Table 1 Numerical Results of problem of 1 with that of (Omar & Adeyeye, 2016; Sunday et al., 2013).

t	**ER**	**CR**	**ENS**	**EOA16**	**ESOA13**
0.100	0.52438528774964299	0.5243852877496429	2.0000e-20	4.9562e-06	5.5744e-12
0.200	0.54758129098202021	0.5475812909820202	3.0000e-20	4.7260e-06	3.9462e-12
0.300	0.56964601178747101	0.5696460117874711	3.0000e-20	8.9799e-06	8.1832e-12
0.400	0.59063462346100907	0.5906346234610091	4.0000e-20	8.5524e-06	3.4361e-11
0.500	0.61059960846429757	0.6105996084642976	5.0000e-20	1.2193e-05	1.9294e-10
0.600	0.62959088965914107	0.6295908896591411	8.0000e-20	1.1608e-05	1.8790e-10
0.700	0.64765595514064328	0.6476559551406433	9.0000e-20	1.4713e-05	1.7768e-10
0.800	0.66483997698218035	0.6648399769821803	9.0000e-20	1.4004e-05	1.7247e-10
0.900	0.68118592418911335	0.6811859241891134	9.0000e-20	1.6643e-05	1.8476e-10
1.000	0.69673467014368329	0.6967346701436833	9.0000e-20	1.5839e-05	3.0058 e-10

See: (Omar & Adeyeye, 2016; Sunday et al., 2013).

Table 2 Numerical Results of problem of problem 2 with that of (Sunday & Agbataobi, 2019)

t	**ER**	**CR**	**ENS**	**ESA19**
0.100	719.87095048413192628	719.87095048413192630000	2.0000e-17	0.0000e00
0.200	746.70631894946328473	746.70631894946328476000	3.0000e-17	0.0000e00
0.300	774.54205699518372529	774.54205699518372527000	2.0000e-17	0.0000e00
0.400	803.41545642515503139	803.41545642515503132000	7.0000e-17	0.0000e00
0.500	833.36519920809658332	833.36519920809658331000	1.0000e-17	0.0000e00
0.600	864.43140930018794572	864.43140930018794562000	1.0000e-16	2.2737e-13

0.700	896.65570639951581410	896.65570639951581398000	1.2000e-16	2.2737e-13
0.800	930.08126170438066714	930.08126170438066698000	1.6000e-16	3.4106e-13
0.900	964.75285575016305883	964.75285575016305864000	1.9000e-16	2.2737e-13
1.000	1000.7169384022341531	1000.7169384022341529000	2.0000e-16	3.4106e-13

See: (Sunday & Agbataobi, 2019).

Table 3 Numerical Results of problem of problem 3 with that of (Omar & Adeyeye, 2016; Badmus et al., 2015)

t	**ER**	**CR**	**ENS**	**EAO16i**	**EAO16ii**	**EBY15**
0.100	0.90483741803595957316	0.90483741803595957288	2.8000e-19	7.5513e-05	9.0730e-12	1.5476e-10
0.200	0.81873075307798185867	0.81873075307798185815	5.2000e-19	6.8684e-05	1.1768e-11	1.3823e-10
0.300	0.74081822068171786607	0.74081822068171786535	7.2000e-19	1.2397e-04	2.3144e-11	1.3282e-10
0.400	0.67032004603563930074	0.67032004603563929987	8.7000e-19	1.1246e-04	2.8440e-11	1.1733e-10
0.500	0.60653065971263342360	0.60653065971263342262	9.8000e-19	1.5237e-04	3.1815e-11	1.1342e-10
0.600	0.54881163609402643263	0.54881163609402643156	1.0700e-19	1.3811e-05	3.4927e-11	9.9385e-11
0.700	0.49658530379140951470	0.49658530379140951357	1.1300e-19	1.6640e-04	3.6582e-11	9.6770e-11
0.800	0.44932896411722159143	0.44932896411722159026	1.1700e-19	1.5076e-04	3.8127e-11	8.4003e-11
0.900	0.40656965974059911188	0.40656965974059911069	1.1900e-19	1.7033e-04	3.8576e-11	8.2517e-11
1.000	0.36787944117144232160	0.36787944117144232039	1.2100e-19	1.5428e-04	3.9020e-11	7.0848e-11

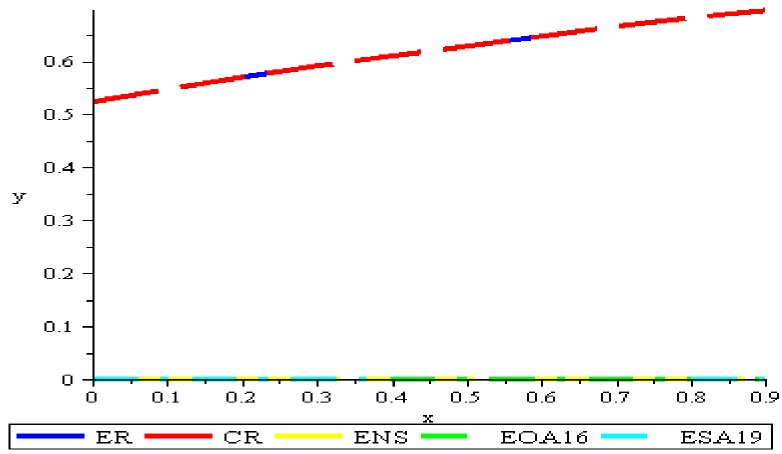

Figure 1 Graphical curves for Problem 1

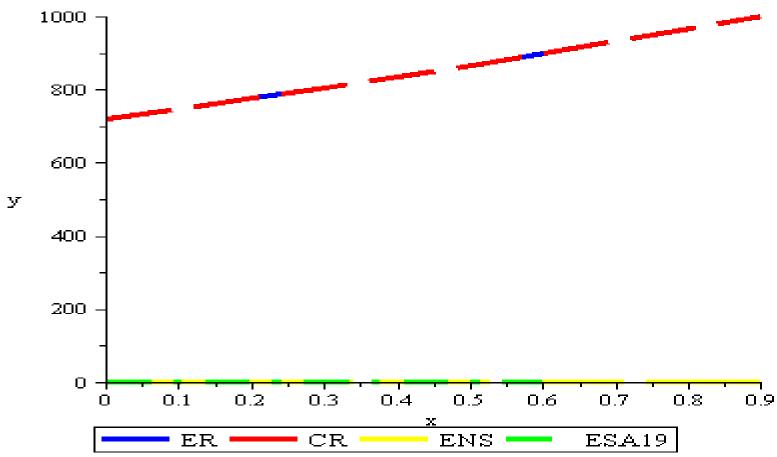

Figure 2 Graphical curves for Problem 17

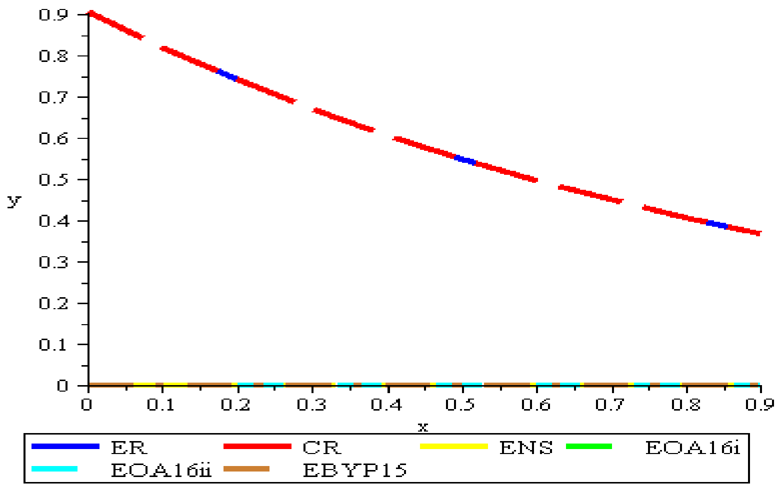

Figure 3 Graphical curves for Problem 3

Discussion

The newly developed scheme meets all criteria for analysis, including order and error constant, consistency, convergence, and zero-stability. This derived scheme was then utilized to address certain epidemic disease problems involving first-order initial value problems of ordinary differential equations. Specifically, Problem 1 was successfully tackled using this approach (Omar & Adeyeye, 2016; Sunday et al., 2013), Referred to as an SIR Model, this epidemiological disease calculates the total number of individuals infected with a transmissible infection within a closed population over a specified duration. Our method, as evidenced by Table 1 and Figure 2, computes more favourably compared to that of (Omar & Adeyeye, 2016; Sunday et al., 2013). Problem 2, known as a Growth Model or bacteria culture, entails growth occurring at a rate proportionate to the current size. This is addressed by (Sunday & Agbataobi, 2019) table 2 clearly demonstrates the convergence of our method across (Sunday & Agbataobi, 2019) additionally, depicted in Figure 2 graphically. Lastly, Problem 3 involves a highly stiff differential equation, which was addressed by (Omar & Adeyeye, 2016; Badmus et al., 2015), where our method outperforms that of (Omar & Adeyeye, 2016; Badmus et al., 2015) as observed in Table 3 and Figure 3.

Conclusion

In this study, a new scheme was employed to simulate an epidemic disease model. This scheme developed using power series polynomials and analyzed numerically, exhibited superior accuracy and faster convergence compared to existing methods explored in the research. Consequently, the new schemes produced favorable outcomes over their predecessors. Moreover, these new schemes demonstrated computational reliability when

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simulating similar epidemic models described by differential equations, surpassing the methods considered previously.

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