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A Note on Free Semigroups

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Abstract
This paper reviews a class of semigroup, the free ones known as free semigroups. The different modes of definition ascribed to free semigroups, concept of freeness, it's subsemigroups are examined. Presentation, constructions and examples of free semigroups and 0-free semigroups are given and properties inherent are presented. External direct products of free semigroups are also reviewed and presented.

Keywords: External direct product**,** Freeness , Free Semigroups, 0-free semigroup

Introduction

A semigroup can be described as a non-empty set S on which an associative operation is defined. S is a semigroup, if the operation φ is associative on it thus: $\forall a, b, c \in S$, $a(bc) = (ab)c$ .That is, φ $(a\ \varphi(b, c))$ = φ $(\varphi(a, b), c)$. Different classes of semigroup according to their abstract nature, structural composition and semigroup with additional restrictions together with their applications have been presented (Abubakar, 2024). One of the classes of semigroup that fall under the abstract nature is the free semigroups. In the wide and growing applications of semigroup theory to automata, languages and machines, it is the free semigroup that play the major role. Free semigroup is a semigroup made of words. The significance in applications of free semigroups is in the fact that the elements are words in an alphabet (Howie, 1995). In this paper, we review some basic definitions of free semigroups, noting the freeness property therein, highlights conditions inherent in it's sub semigroups and examine constructions of free semigroups ,0-free semigroups and their properties and review the necessary and sufficient condition for the direct products of free semigroups to be countably generated.

Preliminaries

In this section, we provide vital definitions and introduce concepts that would be useful to the work. Concepts or definitions not clear to the reader, we refer them to (Clifford &Preston, 1961; Gilbert , 1976; ,Higgins 1992;, Omoelebe &David, 2015; Nwawuru &Udoaka, 2018)

Definition
Word
Consider A to be an alphabet, word on $A = \{x, y\}$ is a countable string of the symbols x, x^{-1}, y , y^{-1} where $x\ and\ x^{-1}$ equally,y and y^{-1} cannot be adjacent because jointly, they form the empty word with the symbol 1.

Remark
If $\propto and\ \beta$ are both words of length m and n, then, their juxtaposition $\propto * \ \beta$ is the word of length $m + n$ obtained by placing $\propto$ to the left of β to produce $\propto \beta$

Definition
A word that has length '0' is termed an empty word and is denoted by $\emptyset\ or\ 1$

Example
If A ={ a, b}*, then a word belonging to A^8 is of the form baabbaba, $A^0 = \emptyset\ or\ 1$ and word belonging to $A^3 = \{ aaa, aab, aba, abb, baa, bab, bba, bbb\}$

Definition
(i)Concatenation (or Juxtaposition)
A computer obtains information from an input terminal and this feeds in a sequence of binary digit symbols of 0's and 1's. The feeding process continues to give the computer a long sequence that is known as concatenation (or juxtaposition) of the two sequences. These inputs process in addition to the binary operation of concatenation form a semigroup that is called the free semigroup which was generated by the input symbols

(ii). Strings of symbols
Any spoken statement, any written statement, any input to a computer is a string of symbols, every Shakespeare play is an element of a non empty set where A is the set consisting of the 52- upper and lower case letters of alphabet in addition to various space and punctuation symbols.

(iii).Coding
By a code (or variable length code) in the alphabet A , we mean a subset P of a set that is not empty set of P that has the property that P is a set of free generators for $\langle P \rangle$.

Remark
There are different types of codes: Error-correcting, Error-detecting and Cyclic codes, prefix codes, biprefix code, suffix codes etc.

Homomorphism
Let $(S,*)$ and $(T,\circ)$ be two binary structures consisting of the sets S and T together with the binary operations * and $\circ$ on S and T respectively. Then a function $f: S \to T$ is said to be a *homomorphism* between $(S,*)$ to (T, o) if, for every $r, s, \in S,$ then

$$f\ (\ r * s) = f(r) \circ f(s)$$

Remarks
(i).If a binary structures is endowed with more than one operation, the homomorphism must preserve all of the operations therein.
(ii)If the binary structures have identities, then, the identities must also be preserved.

Examples
(i).Considering the set of all integers $\mathbb{Z}$ under the given operation of addition '$+$' and the set of positive reals denoted as $R_{>0}$, under multiplication " $\cdot$'. The function $f: \mathbb{Z} \to R_{>0}$ defined by $f(x) = e^x$ is a homomorphism between $(\mathbb{Z}, +) to\ (R_{>0},\ \cdot).$ Under exponents, products of the exponentials $e^x\ and\ e^y$ corresponds to addition of their exponents $x\ and\ y.$

(ii). A vector space is an structure where the underlying sets are vectors. Its operations involves the operation which under addition and for each scalar λ given, a unary operation of multiplication by $\lambda.$ A function $f: S \to T$ between the vector spaces is a homomorphism if it satisfies $f(\ a + b) = f(a) + f(b)$ and $(\lambda a) = \ \lambda\, f(a)$, for all $a, b\ \in S$,for all vectors $a\ and\ b$ in the domain S and all scalars λ given. Such a vector space homomorphism is usually known as a linear transformation.

Isomorphism
If there exists a homomorphism between two structures that is both 1-1 and onto or alternatively, there is a bijective between the two structures ,then, the type of homomorphism exhibited is known as Isomorphism.

Remark
Isomorphic structures possess similar properties and they are not easily differentiable from an algebraic point of view.

Example
Let V_1 and V_2 be two vector defined spaces of the equal finite dimensions over a field $K,$ then, V_1 and V_2 are isomorphic, that is $V_1\ \cong\ V_2$

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Automata

Automata theory is the phenomenon involving abstract machines and automata, as well as the computational analogies that can be resolved using them. It is a theory in theoretical computer science in the area of discrete mathematics (a section of mathematics linking Computer Science). Automata theory is the study of self-operating virtual machines that helps in logical reasoning of input and output exchange, without or involving intermediate stage(s) of computation (or any procedure).

Freeness property

A semigroup S is called free if it is isomorphic to the free semigroup on some given set.

Base of a Semigroup

Base of a semigroup S is expressed as Base (S) = $S \setminus S^2 = \{ x \in S: \forall y, z \in S: x \neq yz\}$ having elements $x \in S$ that are not products of any two or more elements coming from S (Harry, 1996).

Free semigroup

i.A free semigroup is a semigroup having non commutative products where each products can only be uniquely presented

ii. Let M be a non-empty set denoted by M^*.The free semigroup on M is the subsemigroup of M^*containing all the elements of M except the empty string.

iii.Let S be a semigroup . A subset $X \subset S$ is considered to generate S freely if S =$[X]_S$ given that $[X]_S = \bigcup_{n=1}^{\infty} X^n$ and for each map $\alpha_{0:}X \longrightarrow Q$ (where Q is any semigroup) can be extended to a homomorphism.$\alpha: S \longrightarrow Q \ \ni \propto X = \alpha_0$,$\alpha$ is said to be a homomorphism extension of the mapping α_0. If S is said to be freely generated by some subset, then S is a free semigroup.

Examples

i.$(N_+, +)$ is a free semigroup. For the element X = $\{1\}$ can generates it freely: Let $\alpha_{0:} X \longrightarrow P$ be a homomorphism and define $\alpha: N_+ \to P$ by $\alpha\ (n) = \alpha_0\ (1)^n$. Now $\alpha\ X = \alpha_0$ and α presents a homomorphism:

$$\alpha\ (n+m) = \alpha_0\ (1)^{n+m} = \alpha_0\ (1)^n.\ \ \alpha_0\ (1)^m$$
$$= \alpha\ (n). \alpha\ (m) \qquad \text{(Harju, 1996)}$$

ii.Let A represent a non empty set, let A^+ be the set of all countable non empty words $a_1, a_2, a_3, . . ., a_m$ in the alphabet A . Let there exists a juxtaposition on the binary operation to give A^+ in the form $(a_1, a_2, a_3, . . ., a_m)(b_1, b_2, b_3, . . ., b_n) = a_1a_2a_3 . . . a_m\ b_1b_2b_3 . . . b_n$ using this operation, A^+ is a semigroup known as the free semigroup on A .

Remarks

i. Consider P is the generating set for P^+then, P is the unique minimum generating set for P^+ since P =$P^+ \setminus (P^+)^2$.

ii.The map $\alpha: M \longrightarrow M^+$ that associates each a in M with the corresponding one-letter word in M^+ is the referred to as the standard embedding of M in M^+(Higgins, 1992).

Example

Let A = $\{a_1, a_2\}$ be a binary alphabet and consider $S = [a_1 {a_2}^n \mid n \geq 1] = [\, a_1a_2, a_1{a_2}^2, . . .]$.
Base(S)= $S \setminus S^2 = \{\, a_1{a_2}^n \mid n \geq 1]$ and S is freely generated by Base (S)

Free semigroup (Abstract)

F is referred to as a free semigroup on the alphabet A if

i) There exists a map $\alpha: \ A \to \ F$
ii) For each semigroup S and each map ϕ: $A \to S$, we have a unique homomorphism $\psi: F \to S$ and the combination below

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$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & F \arrow[dl, "\psi"] \\ S & \end{tikzcd}$$ commutes.

Remarks

i).Properties are defined (if it exists) up to isomorphism, substituting F and α for S and $\emptyset$ in

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & F \arrow[dl, "\psi"] \\ S & \end{tikzcd}$$

We see that the unique ψ making the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & F' \arrow[dl, "\psi"] \\ F & \end{tikzcd}$$

commute is the identity map 1_F.

ii).If S is a semigroup and A is a generating set for S, then the property α presents a homomorphism ψ from A^+mapped onto S as $S \backsimeq A^+ / \ker \psi$

Example

Monogenic Semigroup

Let S be a semigroup, and consider the index family $\{U_i : i \in I\}$ (with I$\neq \emptyset$) of subsemigroups of S. The intersection of all the subsemigroups U_i, if non empty will bring about subsemigroups of S. For every non-empty subset P of S, we have at least one subsemigroup of S containing P, namely S itself. Therefore, the intersection of all the subsemigroups of S containing P gives a subsemigroup of S containing P denoted by $\langle P \rangle$ that has been confirmed as a subsemigroup of S containing P then $\langle P \rangle \subseteq U$. On the long run, subsequently, $\langle A \rangle = S$ then, P will be a set of generators or a generating set for S.When $\langle p \rangle = \{ p, p^2, p^3, \ . \ . \ . \ \}$.The case scenario of S is a semigroup in which there exists an element a such that $S = \langle p \rangle$ then S is called a monogenic semigroup (Howie,1995).

Free semigroups

Let P be any set and F_P consists of all finite sequence of elements of P. If $(p_1, \ p_2, p_3, . \ . \ ., p_m)$ and $(y_1, y_2, \ . \ . \ ., y_m)$ are elements of F_P ($p \in P, i = 1, 2, \ . \ . \ ., m; \ j = 1\ 2, . \ . \ ., n$), then, defining products using simple juxtaposition $(p_1, \ p_2, p_3, . \ . \ ., p_m)\ (y_1, y_2, . \ . \ ., y_m) \ = (p_1 \ p_2 p_3, . \ . \ . p_m \, y_1 \, y_2 \ . \ . \ . y_m)$. F_p now gives a semigroup called the free semigroup on X.

Remarks

i.Elements of $\mathrm{F_x}$ are called words.

ii. Looking at the element r of R with the sequence (r) having length 1, using definition 2.24 of product

$(r_1, \ r_2, r_3, . \ . \ ., r_m) \ = (r_1)\ (r_2)(\ r_3) . \ . \ . (r_m) = r_1 r_2 r_3 . \ . \ . r_m$

Thus, R is a generator of $\mathrm{F_x}$ (Clifford & Preston, 1961).

Remarks

i. A free semigroup is a semigroup having product that is not commutative where no product can never be expressed in a non complicative terms of other elements therein .

ii.A free semigroup can be described ultimately up to isomorphism by virtue of the number of its alphabet known as rank of free semigroup.

iii. Free semigroups can be classified as free objects in the context of categories..

iv. Free semigroup metamorphosed simply in the theory of automata, the theory that uses coding, formal languages including formal grammar .

v. The free semigroup on P give rise to the subsemigroup P^* which contains all elements excluding the empty string which is designated as P^+.

vi. $P^* = 1 \cup P^+$

Subsemigroups of free semigroup

i. Each subsemigroup $P*$ of a free semigroup P possesses unique irreducible generating set that has elements which is not decomposable into a product on P.

ii.Some subsemigroup of free semigroups are not free

iii.There are some rules on a subsemigroup P^* of free semigroup P are analogous:

a. P presents a free semigroup

b. In each w $\in P$, w $P \cap P \neq \emptyset$ and P w$\cap P \neq \phi \Rightarrow w \in H$

c. For any $w \in$ F, wH $\cap Hw \cap H \neq \emptyset \Rightarrow w \in H$

iv. For unpredictable different words a, b in a free semigroup P, two things possible are for any one of : u and v to be free generators of the subsemigroup that has been generated or there exists a $w \in P \ni u = w^t, v = w^r$ for some natural numbers t and r. The second rule is ascertained if and only if $uv = vu$.

v. The free semigroup existing on an alphabet $P =$ {a,b} contains copies of F_p for any set P of finite cardinality.

Construction and presentation of free semigroup

Presentation of free semigroup

Ruskuc (1995) gave various presentations by way of defining Free semigroups, stressing that even though theoretically each semigroup has a definition given as a presentation which cannot happen for each semigroup to be defined 'nicely' in this way .He reiterated that the most credible class of semigroups for investigation using presentations are the countably presented semigroup with the definition (A/$\mathfrak{R}$) where both A and $\mathfrak{R}$ are countable, and the elements of A are known as the generating character or just generators. The components of $\mathfrak{R}$ are referred to as defining relations in which

$$\mathfrak{R} \subseteq A^+ \times A^+.$$

Example

Given an alphabet P, $P*$ represents all countable words over P. The subsemigroup P^+ represent of all words over P which are not empty, using $P* = P^+ \cup \{\varepsilon\}$ where ε represents empty word. The sets $P*$ and P^+ are transformed into semigroups using products of words known as concatenation. The subsemigroup P^+ is clearly represented which has been freely generated. The smallest conformity on P^+which has the empty set has the asymmetrical relation

{ (w, w) $|$ w $\in P^+$} so, semigroup that has this criteria $\langle \mid P \mid \rangle$ is the free semigroup P^+.

Remark

The demonstration $\langle P \mid \ \rangle$ presents a free semigroup P^+ for the smallest compatibility on P^+ that has the empty set is the asymmetrical relation $\Delta = \{ (w, w) : w \in P^+ \}$. The demonstration $\langle a \mid a^2 = a \rangle$ gives the trivial semigroup which is the same as compatibility depicting full congruence on $\{a\}^+$.

Construction

Let Z be the variety of zero semigroups, P is a non-empty set and S_P be a semigroup defined by

$S_p = S \cup \{0\}$ with multiplication ab = 0 for all a , b $\in S$

Let $i : P \longrightarrow S$ with $i: p \longmapsto p$

Theorem

(P, τ) is the free semigroup on P in Z.

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Proof

Clearly $P \in$ Z. Let T be any zero semigroup and ψ : X⟶ T be any mapping. We want to show t we can have a unique homomorphism $\varphi: P \longrightarrow T$ so that the illustration commutes as:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[drr, "\psi"'] & & P \arrow[d, "\varphi"] \\ & & T \end{tikzcd}$$

Define $\varphi: S \longrightarrow T$ by

$$\varphi\,(x) = \psi\,(x), \forall\, x \in X$$
$$\varphi\,(0) = 0$$

Since $\varphi(xy) = \varphi(0)$

$= 0$

$= \psi\,(x)\;\; \psi\,(y)$

Then, φ is a homomorphism. Moreover, φ is unique, since every homomorphism $\varphi : P \longrightarrow T$ which makes the illustration commutative.

$\psi_{x}| = \varphi_{x}|$, that is $\psi = \varphi$

∴ (P, i) is the free semigroup on P in Z (Aluysiyus,1991).

Imagine P represents arbitrary non-empty set, the set P being an alphabet has letters as its elements. Lets represent by L set of all ordered sequences of countable length bigger than zero and any object of P^* has the feature :$r = (r_1, r_2, r_3, . . . , r_m)$ with m ∈ M and $(x_1, x_2, x_3, . . . , x_m)$ are any elements of P. We usually omit the brackets and the periods in the notation of x, thus abbreviating $x = x_1, x_2, x_3, . . . , x_m$.

The elements of P^* referred to as words (of countable length) on alphabet P. Specifically, word $r = r_1, r_2, r_3, . . . , r_m)$ having length $n \in N$. Two words $t = (t_1, t_2, t_3, . . . , t_s)$ together with $l = (l_1, l_2, l_3, . . . , l_m)$ from P^* intersects if lengths s and m are the same and holds for all $t_i = l_i$ given that i ∈ { 1, 2, . . ., n}. Example if t, l represents two different letters from P, then $v = tltt$ also, $w = ttlt$ represents different words having a cardinality of four. With the direction from above, a one-letter representation will be (t) , as P is a subset of P^***.** Generated P^* is a semigroup, known as free semigroup over P, using the order of concatenation: for $t = (t_1, t_2, t_3, . . . , t_s)$ *and* $l = (l_1, l_2, l_3, . . . , l_m)$ inside P^*,on combination,$tl = t_1 t_2 t_3 . . . t_s\, l_1 l_2 l_3 \;. . . l_m$ the word having length $s + m$ that arises after writing t directly after l.

Example

For $v = tltt$ together with $w = ttlt$ to give $vw = tlttttlt$ and $w = ttlttltt$

Properties of free semigroup

i.Benchmark for freeness

Given that $P \subseteq S$, $x = x_1,\; x_2, x_3, . . . , x_n$ gives a factorization of x over the alphabet X, if each $x_i \in X$. Considering that X is a generator for S, each element $x \in X$ can be has a factored in X. Although, this factorization is not distinct, for this to hold as :

$x_1,\; x_2, x_3, . . . , x_n = x = t_1, t_2, . . . , t_m$

Considering part of $x_i \in X$ (i =1, 2, ... , n) and for some different $t_i \in X$ (i =1, 2, ... , n)

Theorem

A semigroup S is considered as free if and only if each element of S can be distinctly expressed as multiplication of their elements.

Proof

⇒ If $S = P^+$, a free semigroup, then $S \setminus S^2 = P$, and each element has a distinct expression as a multiplication of elements in P.

⇐ Suppose that S has the given property, and denote $S \setminus S^2$ by P**.** We show that S has the defining property of P^+.Let T be a semigroup, and let $\alpha: P \to T$ be an arbitrary map. For each s in S consider the unique expression $s = a_1 a_2 \;. . . \; a_m$ of s as a product of elements in A, and define $\bar{\alpha}: S \longrightarrow T$ by the rule that $s\bar{\alpha} = (a_1\alpha)(a_2\alpha) \;. . . \; (a_m\alpha)$.

If $t = a'_1 a'_2 \ . \ . \ . \ a'_n \ \in S$, then it is clear that the unique expression for st must be $a_1 a_2 \ . \ . \ . \ a_m a'_1 a'_2 \ . \ . \ . \ a'_n$, and from this remark it follows that $\bar{\alpha}$ is a morphism. It is clear that $\overline{\alpha}$ is the unique extension of α to S, and so $S \simeq A^+$, the Free semigroup on A.

Theorem (Universality of free semigroups)
Assume that P^* is the free semigroup over P, given that S is a random semigroup, and $f: P \to S$ be a random map from P into S. Then we have a distinct semigroup homomorphism $\bar{f}: P^* \to S$ that extends f.

Corollary
Each semigroup P presents a homomorphic image to a free one.

Proof.
Let P be a random semigroup; by fixing an alphabet P that has a size of minimum $|S|$ and consider a mapping f linking P onto S. A term homomorphic extension of the function $\bar{f}: P^* \to S$, described by Theorem 3.9 is a mapping of P^* onto S.

Construction of 0-free semigroup (Omoelebele &David, 2015)
Given that P is a language constructed from the alphabet A inside $A^\#$. Then P is called prefix-closed whenever $(\forall\, u \in A^\#, a \in A^+)(ua \in P \Longrightarrow u \in P)$ and P is considered closed under the context of picking non-empty prefixes , or more preferably +-prefix closed , whenever
$(\forall\, u \in A^+, a \in A^+)(ua \in P \Longrightarrow u \in P)$ for the selected alphabet, P^+ -non empty words over $P^+$is referred to as the free semigroup and $A^\#$ as the 0-free semigroups.

The properties of 0-free semigroup $P^\#$are :
i.Its a regular language with the adjoined zero as a word that is,

$$L^\# = (L^+ \cup \{0\}) \subseteq A^\#$$

ii.Its is +-prefixed closed and
iii.It has a unique representation

Proposition (Omoelebele &David, 2015)
The 0-free semigroup $P^\#$ contains a regular language

Axioms for Uniqueness
The uniqueness axiom ascertains that any two free semigroups that have the same base sets of same order are isomorphic and this justifies the term "the free semigroups" on P which is quite different from a free semigroup on P.

Theorem
A semigroup P is freely generated by X if and only if for each $x \in P$, P has a distinct factorization prescribed in X

Proof
Lets first note that the assertion holds for the word semigroups P^+ for which P represents the only smallest generator set. Consider P as an alphabet with the condition that $|P| = |X|$ and allowing $\alpha_0: S \longrightarrow P$ to satisfy bijection condition.
Suppose P is a generator for S freely together with $x \in P$ where
$P = x_1, x_2, x_3, . \ . \ ., x_t = y_1, y_2 , . \ . \ . , y_m \ (x_i, y_i \in X)$
$\alpha(P)$= $\alpha_0(x_1)\alpha_0\ (x_2)\alpha_0(x_3). \ . \ . \ \alpha_0\ (x_t)$= $\alpha_0(y_1)\alpha_0\ (y_2)\alpha_0(y_3). \ . \ . \ \alpha_0\ (y_m)$. In P^+.Since P^+ satisfies the condition of the theorem and $\alpha_0(x_i), \alpha_0(y_i)$ are considered as letters for each i, we must have $\alpha_0(x_i)$ = $\alpha_0(y_i)$ collectively for $i = 1, 2,. \ . \ .$, t(and t = m). Moreover, α_0 is injective, and so x_i= y_i collectively for i bringing about that also S makes certain the claim.

Considering that S assures the uniqueness condition. Denoting $\psi_0 = \alpha^{-1}$ and let $\psi: A^+ \longrightarrow P$ be the homomorphic extension of ψ_0. Now, ψ is surjective, since P generates S. Injectivity is assured, since if $\psi(u) = \psi(v\ )$ for selected $u \neq v \in P^+$ then $\psi(u)$ presents exactly two different factorizations using P. Hence ψ gives an isomorphism, the proof is confirmed.

Necessary and sufficient conditions of external direct products for free semigroups (Nwawuru & Udoaka, 2018)

Let P^+ be free semigroup then, P^+ can be decomposable if there exists a word w $\in P^+$such that $w = a_1 a_2$ for some $a_1, a_2 \in M^+$.
Notation for the set of decomposable free semigroup is represented as $(P^+)^2 = P^+P^+ = \{a_1a_2: a_1, a_2 \in M^+\}$.
The collection of indecomposable free semigroup is generally denoted as $(P^+ \backslash (P^+)^2$.

Example
Let $\{a, ab, aab, \dots\} \in P^+$ be set of words generated by the binary alphabet. The unique indecomposable word will be the one-letter word in P^+.

Remark
Each indecomposable word came out using a generating set. An quick example is the one letter-word "a" which is an indecomposable word in P^+ unfortunately, it goes with the generating set from the binary alphabet. If P is a finite alphabet. Using the abstract analogue of a free semigroup, P^+ is free on A if there is a map $\psi: A \rightarrow P^+$

Lemma
Each semigroup S together with every map $\psi: A \rightarrow S$ presents a unique homomorphism $\vartheta: P^+ \rightarrow S$.

Lemma
Consider P^+ and N^+ to be two finitely generated free semigroups, their external direct product is uniquely defined as $(P\oplus N)^+$ and thus is referred to as finitely generated.

Proof
Consider C and D to be countable alphabet in a way that C and D generates P^+ and N^+ respectively. Representing the external direct product of C and D as $(C \oplus D) = \{(c_1, d_1), (c_2, d_2), . . . (c_n, d_n)\}$and subsequently $w_1 = \{c_1, c_2, . . . , c_n\}$ and $w_2 = \{d_1, d_2, . . . , d_n\}$ are two words formed using the alphabets C and D in a way that mapping φ from $C \oplus D$ to $(P\oplus N)$ gives an epimorphism. This way, we have $(w_1, w_2) \in (C \oplus D)^+$in a way that $\varphi\,(a_1, b_1) = (w_1, w_2)$ for all $(a_1, b_1) \in (C\oplus D)$. Thus, for each word $(w_1, w_2) \in (C\oplus D)^+$ that has been generated using $C\oplus D$.Since $C\oplus D$ is countable , then $(C\oplus D)^+$presents a countably generated free semigroup and this consequential from fact that φ depicts an epimorphism (Nwawuru & Udoaka, 2015)

Thus,
(a) M^+ is finitely generated if A is chosen to be finite
(b) M^+ is finitely presented if the generating set A and the defining relation R can be chosen to be finite, (c) Every indecomposable word belongs to a generating set.
(d) Not every finitely generated free semigroup is finitely presented e.g $\langle a, b | ab\ ia = aba\ (i = 1, 2, 3, \dots\rangle$
(e) A finitely generated free semigroup $\nRightarrow$ finitely presented. (f) $(M^+)^2 \neq M^+ \Longrightarrow (M\oplus N)^+$is not finitely generated (Nwawuru & Udoaka, 2015).

Categorical freeness property
The categorical freeness characteristics of the free semigroup P over the set A is that:
For each P together with a map $\varphi: A \longrightarrow P$,there exists a distinct homomorphism of semigroups $F \longrightarrow P$ that extends φ.

Examples of semigroups that are not free
i.Let $S = \{a, a^2\}$ represents a monogenic semigroup that has index of 2 together with period 1, implying that $a^3 = a^2$. If P generates S, then necessarily $a \in P$ (since two elements multiplies to give a . Consider $\alpha_{0:}\ S \longrightarrow P = (N_+, +)$ be $\ni$ α_0(a) = 1.
If $\alpha_{0:}\ S \longrightarrow P$ gives the extension that α_0, then $\alpha(aa) = \alpha(a)\ \alpha(a) = \alpha_0(a) + \alpha_0(a) = 2$
Similarly $\alpha(aaa) = 3$. However, $a\ a \neq a\ a\ a$ in S and this is a counter informative. Hence, S is not regarded as free.

ii.Let $P = \{r, s, t\}$ depicts an alphabet. The words rs, srs, sr will give a generator of the subsemigroup of P^+. This semigroup $P = \{rs, srs, sr\}$ is not considered free because the element $w = \{srsrs\}$ presents two unequal factorizations in $P: w = sr.srs = srs.rs$

Conclusion

This paper has examined free semigroups and 0-free semigroups construction, presentations and properties noting the uniqueness in each definition given. Presentation, construction and some examples of free semigroups likewise semigroups that are not free are also discussed. The necessary and sufficient conditions whereby external direct products on free semigroup are showcased.

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