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Numerical Application of Enright's Linear Multistep Method on Oscillatory Differential Equations

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Abstract
Oscillatory differential equations play a vital role in modeling natural and physical phenomena in fields such as biology, circuit theory, and fluid dynamics. These equations often lack analytical solutions, necessitating numerical methods for resolution. This study focuses on the development and application of a continuous Enright linear multistep block hybrid method to solve first-order oscillatory differential equations. The method leverages interpolation and collocation techniques to generate a high-order, stable, and accurate numerical scheme. The basic properties, including order, consistency, zero-stability, convergence and the region of absolute stability, are analyzed to validate the method. Numerical experiments are conducted to compare the new method's accuracy and computational efficiency against existing approaches, demonstrating its superior performance in solving oscillatory problems.

Keywords: Oscillatory Differential Equations, Numerical Methods, Enright Linear Multistep Method, Block Hybrid Method, Interpolation and Collocation.

Introduction

Oscillatory differential equations play a crucial role in solving practical problems, as they are widely used to model various natural and physical phenomena (Areo & Edwin, 2020). These equations frequently appear in areas such as biological systems, circuit theory, fluid dynamics, and chemical kinetics. However, they may not always have exact analytical solutions, making it necessary to employ numerical methods for their resolution (Skwame et al., 2017).

Empirical computations are essential for exploring the behavior of such systems. To illustrate this, the first-order oscillatory differential equations can be sampled as follows:

$$y'(x) = f(x, y), \; y(x_0) = \chi_0 \tag{1}$$

The block hybrid method, built on the foundation of the Enright Linear Multistep Method, is a powerful numerical approach for solving ordinary differential equations (ODEs). The Enright method, a member of the linear multistep family, employs multiple steps to approximate the solution over an interval by using past computed points, which enhances accuracy and stability (Ayinde et al., 2022).

The block hybrid method using the Enright framework is particularly advantageous in solving higher-order ODEs and stiff systems due to its ability to maintain stability over larger step sizes (Shokri & Shokri, 2028). Its block structure enables parallel computation, where the solution is evaluated at several nodes within an interval simultaneously, making it suitable for modern computational architectures. This approach reduces the need for iterative corrections, as solutions at intermediate points are inherently embedded in the method. As a result, the block hybrid method delivers superior performance in terms of accuracy, stability, and computational efficiency, making it an essential tool in the numerical analysis of differential equations across various scientific and engineering applications (Raymond et al., 2023). Numerous numerical methods have been developed for solving Equation (1), by different researchers among others are Ayinde et al. (2021), Kida et al. (2022), Omar and Adeyeye, (2016), Ayinde et al. (2022), Oyedepo et al. (2022), Oyedepo et al. (2023), Oyedepo et al. (2024).

In this study, a continuous formulation of the first-order linear multistep method will be developed using the Enright linear multistep collocation technique. The resulting method will generalize the Enright method and include other potential variants. The Enright block method will be applied to oscillatory differential equations, enabling the simultaneous computation of numerical solutions. Similar to implementations discussed by James et al. (2013), Sunday et al. (2015b, 2015a, 2013).

Formulation of the Method

The formulation of the new method was influenced by the application of interpolation and collocation techniques, drawing inspiration from the foundational methodologies outlined by Skwame et al. (2017). Consider the general linear multistep method of the form

$$\sum_{j=0}^{K} \alpha_j y_{n+j} = h\sum_{j=0}^{K} \beta_j f_{n+j} \tag{2}$$

The generalized Enright's formula for solving first order nonlinear equation of the form (1.1) using one-step linear multistep method is of the form

$$\sum_{j=0}^{1} \alpha_j y_{n+j} = h\sum_{j=0}^{1} \beta_j f_{n+j} \tag{3}$$

Where

$y_{n+j} = y(x_{n+jh})$ and $f_{n+j} = f(x_{n+jh},\ \ y(x_{n+jh}))$

x_n is a discrete point at x, α_j and β_j are coefficients to be determined. To obtained the method of the form (3), $y(x)$ is approximated by a basis polynomial of the form

$$y(x) = \sum_{j=0}^{\Omega} \alpha_j \left(\frac{x - x_n}{h}\right)^j \tag{4}$$

equation (4) will be used for the derivation of the main and complementary methods for the class of Enright's method which is a special case of (4). Now interpolating (4) at point α_{n+1} and collocating the first derivatives of (4) at points $\beta_n, \beta_{n+\frac{1}{4}}, \beta_{n+\frac{1}{3}}, \beta_{n+\frac{1}{2}}, \beta_{n+\frac{2}{3}}, \beta_{n+\frac{3}{4}}, \beta_{n+1}$.

$$\left.\begin{array}{l} y(x) = \alpha_1 \\ y'(x) = \beta_j \end{array}\right\} j = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$$

The system of equations generated are solved to obtained the coefficients α_1 and β_j which are used to generate the continuous multistep method of Enright of the form

$$y(x) = \alpha_1 y_{n+1} + h\sum_{j=0}^{k} \beta_j f_{n+j} \tag{5}$$

we obtain a system of equation represented in matrix form

$$\begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} & 7x^6_{n+\frac{1}{4}} \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2_{n+\frac{1}{3}} & 4x^3_{n+\frac{1}{3}} & 5x^4_{n+\frac{1}{3}} & 6x^5_{n+\frac{1}{3}} & 7x^6_{n+\frac{1}{3}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} & 5x^4_{n+\frac{3}{4}} & 6x^5_{n+\frac{3}{4}} & 7x^6_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} \tag{6}$$

Applying the Gaussian elimination method on Equation (6) gives the coefficient $a_i's, \quad for \quad i = 0(1)8$.

These values are then substituted into Equation (4) to give the implicit continuous hybrid method of the form:

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$$y(x)=\alpha_1(x)y_{n+1}+h[\beta_i(x)f_{n+i}],\ i=0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1 \qquad (7)$$

where the values of the continuous schemes $\alpha_1,\ \beta_0,\ \beta_{\frac{1}{4}},\ \beta_{\frac{1}{3}},\ \beta_{\frac{1}{2}},\ \beta_{\frac{2}{3}},\ \beta_{\frac{3}{4}},\ \beta_1$ in (7) are

1

$\frac{48}{7}ht^7$ ✍ $28ht^6$ 🖃 $\frac{707}{15}ht^5$ ✍ $\frac{1015}{24}ht^4$ 🖃 $\frac{196}{9}ht^3$ ✍ $\frac{77}{12}ht^2$ 🖃 ht ✍ $\frac{151}{2520}h$

✍ $\frac{6144}{35}ht^7$ 🖃 $\frac{3328}{5}ht^6$ ✍ $\frac{15104}{15}ht^5$ 🖃 $768ht^4$ ✍ $\frac{13568}{45}ht^3$ 🖃 $\frac{256}{5}ht^2$ ✍ $\frac{256}{315}h$

$\frac{11664}{35}ht^7$ ✍ $\frac{6156}{5}ht^6$ 🖃 $\frac{8991}{5}ht^5$ ✍ $\frac{10449}{8}ht^4$ 🖃 $\frac{4779}{10}ht^3$ ✍ $\frac{729}{10}ht^2$ 🖃 $\frac{243}{280}h$

✍ $\frac{2304}{7}ht^7$ 🖃 $1152ht^6$ ✍ $\frac{7856}{5}ht^5$ 🖃 $1048ht^4$ ✍ $\frac{1040}{3}ht^3$ 🖃 $48ht^2$ ✍ $\frac{104}{105}h$

$\frac{11664}{35}ht^7$ ✍ $\frac{5508}{5}ht^6$ 🖃 $\frac{7047}{5}ht^5$ ✍ $\frac{7047}{8}ht^4$ 🖃 $\frac{1377}{5}ht^3$ ✍ $\frac{729}{20}ht^2$ 🖃 $\frac{243}{280}h$

✍ $\frac{6144}{35}ht^7$ 🖃 $\frac{2816}{5}ht^6$ ✍ $\frac{10496}{15}ht^5$ 🖃 $\frac{1280}{3}ht^4$ ✍ $\frac{5888}{45}ht^3$ 🖃 $\frac{256}{15}ht^2$ ✍ $\frac{256}{315}h$

$\frac{48}{7}ht^7$ ✍ $20ht^6$ 🖃 $\frac{347}{15}ht^5$ ✍ $\frac{107}{8}ht^4$ 🖃 $\frac{71}{18}ht^3$ ✍ $\frac{1}{2}ht^2$ ✍ $\frac{151}{2520}h$

Evaluate the (7) using Enright's linear multistep method to gives the new method as

$$\left.\begin{aligned}
y_{n+\frac{1}{4}}-y_n &= \frac{41059}{645120}hf_n+\frac{3313}{5040}hf_{n+\frac{1}{4}}-\frac{51273}{71680}hf_{n+\frac{1}{3}}+\frac{1357}{3360}hf_{n+\frac{1}{2}}-\frac{4131}{14336}hf_{n+\frac{2}{3}}+\frac{667}{5040}hf_{n+\frac{3}{4}}-\frac{2411}{645120}hf_{n+1}\\
y'_{n+\frac{1}{3}}-y_n &= \frac{12967}{204120}hf_n+\frac{5888}{8505}hf_{n+\frac{1}{4}}-\frac{1667}{2520}hf_{n+\frac{1}{3}}+\frac{3368}{8505}hf_{n+\frac{1}{2}}-\frac{143}{504}hf_{n+\frac{2}{3}}+\frac{3328}{25515}hf_{n+\frac{3}{4}}-\frac{251}{68040}hf_{n+1}\\
y_{n+\frac{1}{2}}-y_n &= \frac{2573}{40320}hf_n+\frac{212}{315}hf_{n+\frac{1}{4}}-\frac{2511}{4480}hf_{n+\frac{1}{3}}+\frac{52}{105}hf_{n+\frac{1}{2}}-\frac{1377}{4480}hf_{n+\frac{2}{3}}+\frac{44}{315}hf_{n+\frac{3}{4}}-\frac{157}{40320}hf_{n+1}\\
y_{n+\frac{2}{3}}-y_n &= \frac{541}{8505}hf_n+\frac{17408}{25515}hf_{n+\frac{1}{4}}-\frac{184}{315}hf_{n+\frac{1}{3}}+\frac{5056}{8505}hf_{n+\frac{1}{2}}-\frac{13}{63}hf_{n+\frac{2}{3}}+\frac{1024}{8505}hf_{n+\frac{3}{4}}-\frac{92}{25515}hf_{n+1}\\
y_{n+\frac{3}{4}}-y_n &= \frac{4563}{71680}hf_n+\frac{381}{560}hf_{n+\frac{1}{4}}-\frac{41553}{71680}hf_{n+\frac{1}{3}}+\frac{657}{1120}hf_{n+\frac{1}{2}}-\frac{2187}{14336}hf_{n+\frac{2}{3}}+\frac{87}{560}hf_{n+\frac{3}{4}}-\frac{267}{71680}hf_{n+1}\\
y_{n+1}-y_n &= \frac{151}{2520}hf_n+\frac{256}{315}hf_{n+\frac{1}{4}}-\frac{243}{280}hf_{n+\frac{1}{3}}+\frac{104}{105}hf_{n+\frac{1}{2}}-\frac{280}{243}hf_{n+\frac{2}{3}}+\frac{256}{315}hf_{n+\frac{3}{4}}-\frac{151}{2520}hf_{n+1}
\end{aligned}\right\} \qquad (8)$$

Analysis of Basic Properties of the Method

The fundamental characteristics of the newly developed methods for solving nonlinear initial value problems of the form (1) have been examined to confirm their validity, as discussed by Skwame et al. (2017). Key properties analyzed include order, error constant, consistency, and zero-stability, which collectively determine the methods' convergence behavior. Additionally, the region of absolute stability for these methods has been derived in this section.

Order and Error Constant of the new method

Definition 1: Consider the linear operator associated with the new method be defined as

$$L[y(x);h]=\sum_{j=0}^{k}\left\{\alpha_j y(x_n+jh)-\alpha_{vi}y(x_n+vih)-h^d\beta_j y^d(x_n+jh)-h^d\beta_{vi}y^d(x_n+vih)\right\}$$

(9)

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. We expand $y(x_n+jh)$ and $y^d(x_n+jh)$ as a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x);h\}=C_0y(x)+C_1y'(x)+\ldots+C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\ldots$$

(10)

The new method and the associated linear difference operators are said to have order p if $C_0=C_1=\ldots=C_p=C_{p+1}=C_{p+2}=0,\ C_{p+3}\neq 0$. The order is also defined as the largest positive real number p

that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution, the term C_{p+3} is called the error constant and implies that the local truncation error for the new method is given by,

$$T_{n+k} = C_{p+3} h^{p+3} y^{p+3}(t) + O(h^{p+4})$$

(11)

By definition (1), we compute the new method as

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{3313}{5040}\left(\frac{1}{4}\right) - \frac{51273}{71680}\left(\frac{1}{3}\right) + \frac{1357}{3360}\left(\frac{1}{2}\right) - \frac{4131}{14336}\left(\frac{2}{3}\right) + \frac{667}{5040}\left(\frac{3}{4}\right) - \frac{2411}{645120}(1) \right] \\ y\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{5888}{8505}\left(\frac{1}{4}\right) - \frac{1667}{2520}\left(\frac{1}{3}\right) + \frac{3368}{8505}\left(\frac{1}{2}\right) - \frac{143}{504}\left(\frac{2}{3}\right) + \frac{3328}{25515}\left(\frac{3}{4}\right) - \frac{251}{68040}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j}{j!} - y_n \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{212}{315}\left(\frac{1}{4}\right) - \frac{2511}{4480}\left(\frac{1}{3}\right) + \frac{52}{105}\left(\frac{1}{2}\right) - \frac{1377}{4480}\left(\frac{2}{3}\right) + \frac{44}{315}\left(\frac{3}{4}\right) - \frac{157}{40320}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j}{j!} - y_n \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{17408}{25515}\left(\frac{1}{4}\right) - \frac{184}{315}\left(\frac{1}{3}\right) + \frac{5056}{8505}\left(\frac{1}{2}\right) - \frac{13}{63}\left(\frac{2}{3}\right) + \frac{1024}{8505}\left(\frac{3}{4}\right) - \frac{92}{25515}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{3}{4}\right)^j}{j!} - y_n \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{381}{560}\left(\frac{1}{4}\right) - \frac{41553}{71680}\left(\frac{1}{3}\right) + \frac{657}{1120}\left(\frac{1}{2}\right) - \frac{2187}{14336}\left(\frac{2}{3}\right) + \frac{87}{560}\left(\frac{3}{4}\right) - \frac{267}{71680}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(1)^j}{j!} - y_n \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j \left[\frac{256}{315}\left(\frac{1}{4}\right) - \frac{243}{280}\left(\frac{1}{3}\right) + \frac{104}{105}\left(\frac{1}{2}\right) - \frac{280}{243}\left(\frac{2}{3}\right) + \frac{256}{315}\left(\frac{3}{4}\right) - \frac{151}{2520}(1) \right] \end{bmatrix} = 0$$

(12)

The new method is of uniform order 6 with error constant given by

$$C_{p+1} = \left[1.4002\times10^{-08} \quad 1.3861\times10^{-08} \quad -9.1571\times10^{-07} \quad 1.3861\times10^{-08} \quad 1.4002\times10^{-08} \quad 1.3861\times10^{-08}\right].$$

Consistency of the new method

According to the numerical analyst, the new method is said to be consistent if it satisfies one of the following conditions;

(i) the order $p \geq 1$,

(ii) $\sum_{j=0}^{k} \alpha_j = 0$, and

(iii) $\rho'(1) = \sigma(1)$

Hence, the new method is consistent since it has order $p \geq 1$.

Zero-Stability the new method

Definition 2: The new method is said to be zero-stable if no roots Z_s, $s = 1, \ldots, n$ of the first characteristic polynomial $\bar{\rho}(z)$ is define by $\bar{\rho}(z) = \det\left[z\bar{A} - \bar{E}\right]$ has modulus greater than one, i.e $|Z_s| \leq 1$ and every roots with $|Z_s| = 1$ has multiplicity not exceeding two. That is if the roots z_s, $s = 1, 2, \ldots, n$ of the first characteristic polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det\left[zA^{(0)} - E\right]$$

(13)

Applying definition 2, the first characteristic polynomial is given by,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - z\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1) \tag{14}$$

gives $z = 0, 0, 0, 0, 0, 1$. Hence, the new method is zero-stable.

Convergence of the new Method

According to numerical analyst, the new method is said to be convergent since it is consistent and zero-stable.

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Region of Absolute Stability of the new Method

Definition 3: The new method is said be absolutely stable in the region of the complex plane if, for all $h(\lambda h) \in \Re$, all roots of the stability polynomial $\pi(r,h)$ associated with the method satisfy $|r_s| < 1,\ s = 1,2,\cdots,k \ and\ |r_s| < |r_1|, s = 2,3,\cdots,k$.

Applying the definition 3 on the new method using the boundary locus method, we obtain the stability polynomial of new method as

$$\bar{h}(w) = \left(-\frac{1}{241920}w^5 - \frac{1}{241920}w^6\right)h^{12} + \left(-\frac{11}{103680}w^5 - \frac{11}{103680}w^6\right)h^{10} + \left(-\frac{19}{10368}w^5 + \frac{7}{4320}w^6\right)h^8$$
$$+\left(-\frac{29}{1728}w^5 - \frac{29}{1728}w^6\right)h^6 + \left(-\frac{65}{432}w^4 + \frac{101}{864}w^5\right)h^4 + \left(-\frac{1}{2}w^5 - \frac{1}{2}w^6\right)h^2 - 2w^4 + w^5$$

(15)

Using the stability polynomial (15), the region of absolute stability of the new method is shown in Figure1 as

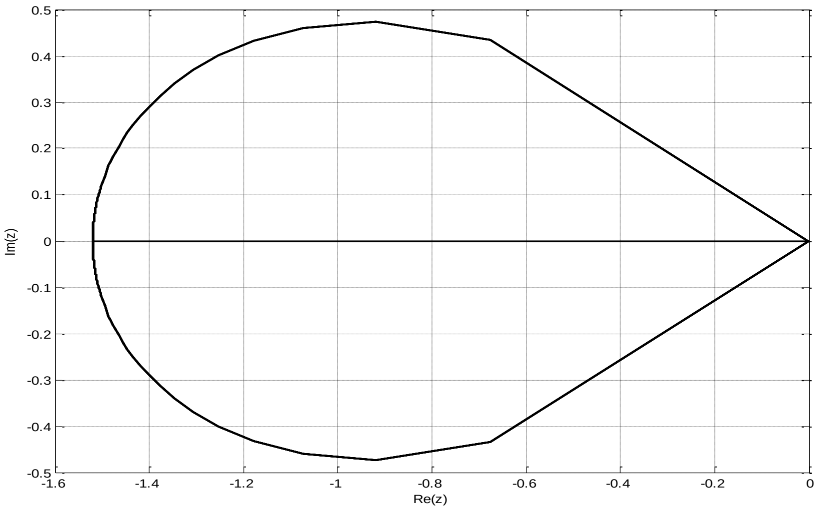

Figure1: Stability region of the new method.

The stability region obtained in Figure 1 is $A-stable$.

Numerical Experiments

The newly developed Enright Block linear multistep method will be applied to several nonlinear first-order initial value problems of ordinary differential equations in the form (2) as outlined below. The outcomes are presented both graphically and in tabular format for clarity and comparison. The following notation will be used in the tables and figures.

ES means Exact Solution

CSNM means Computed Solution of New Method

ENM means Error in new Method

ESE15a means Sunday et al. (2015a)

ESE13 means Sunday et al. (2013)

EJE13 means Error in James et al. (2013)

ESE15b means Sunday et al. (2015b)

Problem 1: We consider the oscillatory differential equation solved by [Sunday et al. (2015a), Sunday et al. (2013)] is given by

$$\frac{dy}{dx} = -\sin(x) - 200(y - \cos(x)),\ h = 0.01,\ y(0) = 0 \quad (16)$$

with the exact solution

$$y(x) = \cos(x) - e^{-200x} \quad (17)$$

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Table 1: Showing the result for oscillatory differential equation (16) with that of (Sunday et al., 2015a; Sunday et al., 2013)

x	**ES**	**CSNM**	**ENM**	**ESE15a**	**ESE13**
0.001	0.18126874692477177712	0.18126874692477177712	0.0000e00	3.7249e-10	6.5812e-06
0.002	0.32967795396412439246	0.32967795396412439246	0.0000e00	5.2169e-10	2.9379e-06
0.003	0.45118386391042716158	0.45118386391042716158	0.0000e00	6.7870e-10	9.3961e-06
0.004	0.55066303589223450724	0.55066303589223450724	0.0000e00	7.6010e-10	1.1305e-05
0.005	0.63210805885482676508	0.63210805885482676508	0.0000e00	7.4126e-10	7.9107e-06
0.006	0.69878778814058064233	0.69878778814058064233	0.0000e00	7.4495e-10	1.0313e-05
0.007	0.75337853615825529977	0.75337853615825529977	0.0000e00	7.2211e-10	1.0426e-05
0.008	0.79807148217492301264	0.79807148217492301264	0.0000e00	6.5649e-10	7.7981e-05
0.009	0.83466061205144457875	0.83466061205144457875	0.0000e00	6.1326e-10	8.4900e-05
0.01	0.86461471717914105002	0.86461471717914105002	0.0000e00	5.6367e-10	8.0388e-05

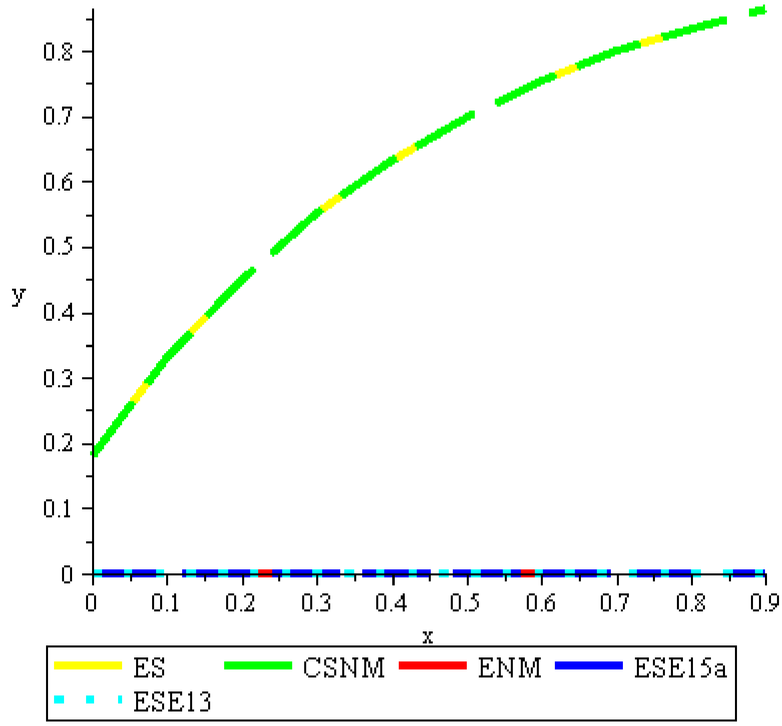

Figure 2: textual curve of table 1 when solving problem 1

Example 2: Examining the oscillatory differential equation

$$y' = xy,\ y(0) = 1,\ 0 \le x \le 1,\ h = 0.1 \qquad (18)$$

Which possesses a solution that can be expressed analytically

$$y(x) = \exp\left(\frac{1}{2}x^2\right) \qquad (19)$$

Source: [James et al. (2013); Sunday et al. (2015b)]

Table 2: Showing the result for oscillatory differential equation (17) with that of [James et al. (2013), Sunday et al. (2015b)]

x	ES	**CSNM**	ENM	EJE13	ESE15b
0.1	1.00501252085940106340	1.00501252085940106330	1.0000e-19	1.6554e-11	1.2473e-13
0.2	1.02020134002675581020	1.02020134002675581010	1.0000e-19	4.3981e-11	2.4989e-13
0.3	1.04602785990871694270	1.04602785990871694260	1.0000e-19	7.8451e-11	4.0149e-13
0.4	1.08328706767495855440	1.08328706767495855450	1.0000e-19	1.2662e-11	5.7196e-13
0.5	1.13314845306682631680	1.13314845306682631690	1.0000e-19	1.9709e-10	7.5116e-13
0.6	1.19721736312181016490	1.19721736312181016500	1.0000e-19	3.0180e-10	9.2698e-13

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0.7	1.27762131320488661070	1.27762131320488661080	1.0000e-19	4.5771e-10	3.0572e-12
0.8	1.37712776433595708450	1.37712776433595708500	5.0000e-19	6.8954e-09	3.1135e-12
0.9	1.49930250005676686970	1.49930250005676687010	4.0000e-19	1.0336e-09	6.1995e-12
1.0	1.64872127070012814680	1.64872127070012814750	7.0000e-19	1.5435e-09	6.6348e-12

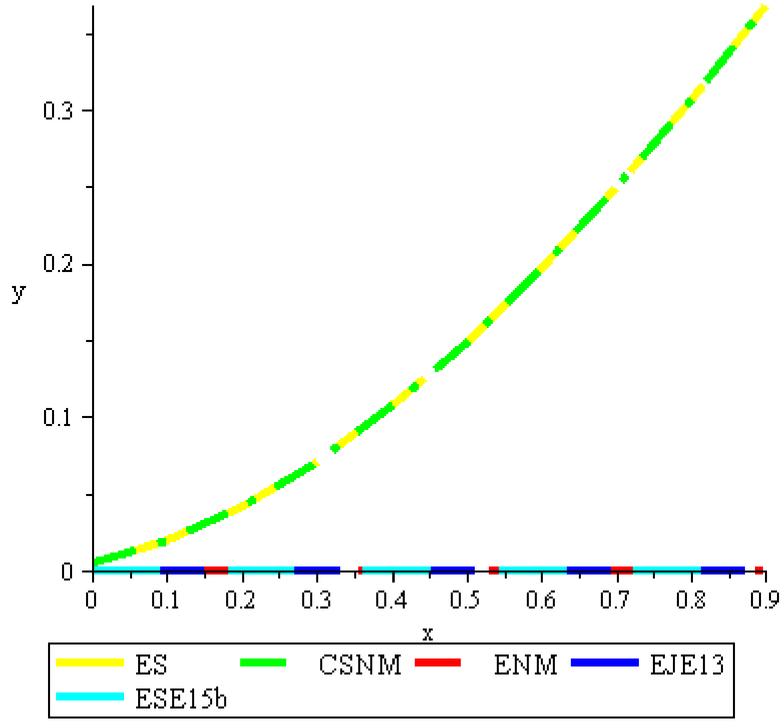

Figure 3: textual curve of table 2 when solving problem 2

Problem 3: James et al. (2013) and Sunday et al. (2015b) investigate the highly stiff oscillatory differential equation presented below:

$$y' = x - y,\ y(0) = 0,\ 0 \le x \le 1 \tag{20}$$

Whose exact solution is given as

$$y(x) = x + \exp(-x) - 1 \tag{21}$$

Table 3: Showing the result oscillatory differential equation (20) with James et al. (2013), Sunday et al. (2015b)

x	ES	CSNM	ENM	ESE15b	EJE13
0.1	0.00483741803595957320	0.00483741803595958356	1.0360e-17	1.0899e-14	1.6554e-11
0.2	0.01873075307798185870	0.01873075307798187748	1.8780e-17	3.6577e-14	4.3981e-11
0.3	0.04081822068171786610	0.04081822068171789159	2.5490e-17	4.4761e-14	7.8451e-11
0.4	0.07032004603563930070	0.07032004603563933154	3.0840e-17	6.1209e-14	1.2662e-11
0.5	0.10653065971263342360	0.10653065971263345843	3.4830e-17	6.1209e-14	1.9709e-10
0.6	0.14881163609402643260	0.14881163609402647045	3.7850e-17	7.0592e-14	3.0180e-10

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0.7	0.19658530379140951470	0.19658530379140955462	3.9920e-17	7.9268e-14	4.5771e-10
0.8	0.24932896411722159140	0.24932896411722163269	4.1290e-17	8.3601e-15	6.8954e-09
0.9	0.30656965974059911190	0.30656965974059915385	4.1950e-17	9.4146e-15	1.0336e-09
1.0	0.36787944117144232160	0.36787944117144223159	9.0010e-17	9.7071e-15	1.5435e-09

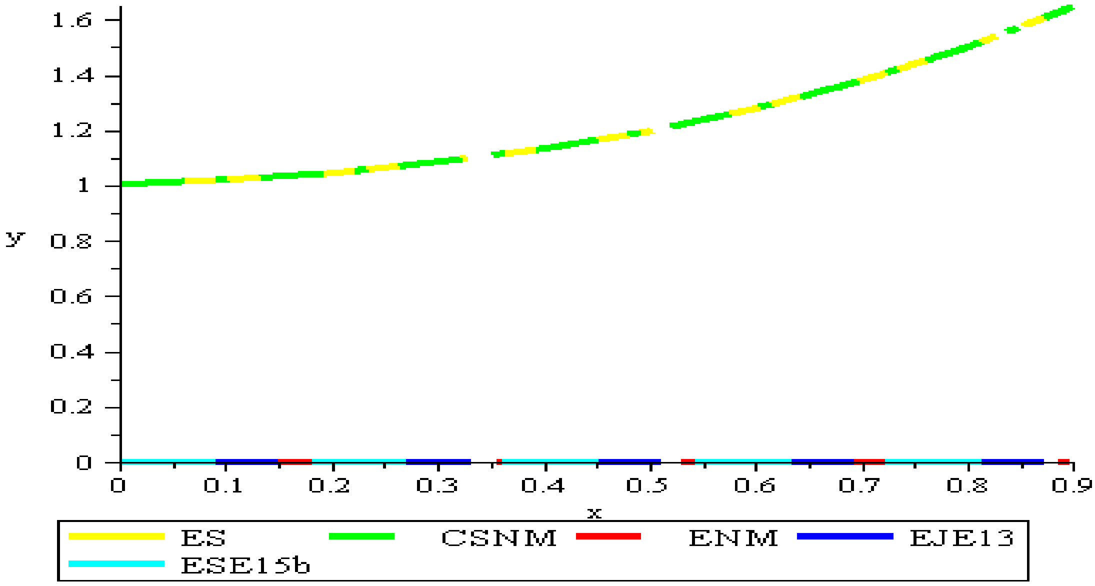

Figure 4: textual curve of table 3 when solving problem 3

Discussion

The results of the problems solved using the new Enright block linear multistep method are presented comprehensively in tables and figures. This approach provides both numerical evidence and visual confirmation of the method's accuracy, efficiency and stability across various types of oscillatory differential equations. Table 1 highlights the exact solution, computed solution of the new method, and the errors (ENM) for problem 1. The error (ENM) reveals that the new method consistently produced zero error at each step, emphasizing its precision. This zero error indicates a perfect alignment between the computed solution and the exact solution. Figure 2 visually compares the CSNM and ES across the domain. The graph shows an exact overlap, with the CSNM perfectly tracing the ES curve. This visual confirmation reinforces the numerical results, showcasing the reliability of the method in solving non-stiff oscillatory problems.

Table 2 summarizes the performance of the new method alongside comparative results from previous methods. The ENM column demonstrates a consistent zero error for the new method, in contrast to the errors observed in the other methods. This improvement underscores the method's robustness and superior handling of oscillatory behavior. Figure 3 provides a graphical representation of the results. The figure shows a seamless overlap between the CSNM and ES, confirming the method's ability to produce highly accurate solutions. The deviation of prior methods from the ES is also apparent, highlighting the new method’s enhanced accuracy.

Table 3 presents the solutions for a problem characterized by high-frequency oscillations. The new method maintains zero error across the solution domain, even in regions of rapid oscillations. In contrast, prior methods struggle to achieve similar accuracy, as reflected in their higher error values. The graphical comparison in Figure 4 vividly demonstrates the effectiveness of the block hybrid method. The CSNM perfectly aligns with the ES, while the solutions from earlier methods show deviations, especially in regions with intensified oscillations.

Conclusion

The Enright linear multistep method was derived by applying interpolation and collocation techniques to construct numerical approximations for solving differential equations. The method employs multiple steps to generate solutions, leveraging past computed values to improve accuracy and efficiency. Its derivation ensures consistency, zero-stability, and convergence, making it suitable for addressing various types of differential equations, including oscillatory problems. The analysis of the method focused on its stability properties, order of accuracy, and computational efficiency. The stability regions were evaluated, demonstrating its capability to

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handle stiff equations under specific conditions. Moreover, the method's theoretical framework confirmed its reliability and effectiveness in approximating solutions. Results from the implementation of the Enright method revealed moderate accuracy when solving oscillatory differential equations. However, comparative evaluations showed that while the method performs adequately, it is less effective in preserving oscillatory behaviors compared to advanced techniques like the block hybrid method. This highlights the need for further refinements to enhance its performance in specialized applications.

References

Areo, E. A., & Edwin, O. A. (2020). Multi-derivative multistep method for initial value problems using boundary value technique. *Open Access Library Journal, 7*, e6063. https://doi.org/10.4236/oalib.1106063

Ayinde, A. M., Ishaq, A. A., Latunde, T., & Sabo, J. (2021). Efficient numerical approximation methods for solving high-order integro-differential equations. *Caliphate Journal of Science & Technology, 2*(3), 188-195. https://doi.org/10.4314/cajost.v2i3.4

Ayinde, A. M., James, A. A., Ishaq, A. A., & Oyedepo, T. (2022). A new numerical approach using Chebyshev third kind polynomial for solving integro-differential equations of higher order. *Gazi University Journal of Science Part A: Engineering and Innovation, 9*(3), 259-266. https://www.ajol.info/index.php/cajost/article/view/218025/205619

James, A. A., Adesanya, A. O., & Fasasi, M. K. (2013). Starting order seven method accurately for the solution of IVPs of first order ODEs. *Progress in Applied Mathematics, 6*(1), 30-39. https://doi.org/10.3968/j.pam.1925252820130601.5231

Kida, M., Adamu, S., Aduroja, O. O., & Pantuvo, T. P. (2022). Numerical solution of stiff and oscillatory problems using third derivative trigonometrically fitted block method. *Journal of Nigerian Society of Physical Sciences, 4*(1), 34-48. https://doi.org/10.46481/jnsps.2022.196

Omar, Z., & Adeyeye, A. (2016). Numerical solution of first order initial value problems using a self-starting implicit two-step Obrechkoff-type block method. *Journal of Mathematics and Statistics, 12*(2), 127-134. https://doi.org/10.3844/jmssp.2016.127.134

Oyedepo, T., Ayinde, A. M., & Didigwu, E. N. (2024). Vieta-Lucas polynomial computational technique for Volterra integro-differential equations. *Electronic Journal of Mathematical Analysis and Applications, 12*(1), 1-8. https://doi.org/10.21608/ejmaa.2023.232998.1064

Oyedepo, T., Ayoade, A. A., Ajileye, G., & Ikechukwu, N. J. (2023). Legendre computational algorithm for linear integro-differential equations. *Cumhuriyet Science Journal, 44*(3), 561-566. https://doi.org/10.17776/csj.1267158

Oyedepo, T., Taiwo, O. A., Adewale, A. J., Ishaq, A. A., & Ayinde, A. M. (2022). Numerical solution of system of linear fractional integro-differential equations by least squares collocation Chebyshev technique. *Mathematics and Computational Sciences*, *3*(2), 10-21. https://mcs.qut.ac.ir/article_252248_6d819941cc8ce8de39705266c8ed036b.pdf

Raymond, D., Kyagya, T. Y., Sabo, J., & Lydia, A. (2023). Numerical application of higher order linear block scheme on testing some modeled problems of fourth order problem. *African Scientific Reports, 2*(1), 23-38. https://asr.nsps.org.ng/index.php/asr/article/download/76/35

Shokri, A., & Shokri, A. A. (2013). The new class of implicit L-stable hybrid Obrechkoff method for the numerical solution of first order initial value problems. *Computers & Physics Communications, 184*, 529-530. https://doi.org/10.1016/j.cpc.2012.09.023

Skwame, Y., Sabo, J., & Kyagya, T. Y. (2017). The constructions of implicit one-step block hybrid methods with multiple off-grid points for the solution of stiff differential equations. *Journal of Scientific Research and Reports, 16*(1), 1-7. https://doi.org/10.9734/JSRR/2017/34986

Sunday, J., James, A. A., Odekunle, M. R., & Adesanya, A. O. (2015a). Chebyshevian basis function-type block method for the solution of first-order initial value problems with oscillating solutions. *Journal of Mathematics and Computer Science, 5*, 463-490. https://www.scik.org/index.php/jmcs/article/download/2114/1151

Sunday, J., Odekunle, M. R., Adesanya, A. O., & James, A. A. (2013). Extended block integrator for first-order stiff and oscillatory differential equations. *American Journal of Computational and Applied Mathematics, 3*, 283-290. https://doi.org/10.5923/j.ajcam.20130306.03

Sunday, J., Skwame, Y., & Tumba, P. (2015b). A quarter-step hybrid block method for first-order ordinary differential equations. *Journal of Advances in Mathematics and Computer Science, 6*(4), 269-278. https://doi.org/10.9734/BJMCS/2015/14139

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