FNAS Journal of Mathematical Modeling and Numerical Simulation
Print ISSN: 3027-1282
www.fnasjournals.com
Volume 2; Issue 2; March 2025; Page No. 19-26.

Application of the Taylor Series Technique to the Solution of Bratu Problems

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Abstract
This study presents a numerical solution of the Bratu differential equations (BDE) using Taylor's series technique. The effectiveness and reliability of the proposed method were further demonstrated by two numerical examples. The outcomes were also compared to other previously published research. Our suggested approach outperforms the Salem and Thanoon (2022) method in terms of approximating the exact solution. The Maple 18 software was used to perform the computations.

Keywords: Non-linear differential equations, Taylor series, Bratu equations, Approximate solution.

Introduction

Consider the nonlinear differential equation of the form:

$$Numerator\,\frac{d^2w}{dx^2} + \gamma e^{\alpha w} = 0; \quad 0 < x < 1; \quad \gamma, \alpha \in \Re \tag{1}$$

with initial conditions $w(0) = w'(0) = 0$. (2)

Equation (1) often known as a one-dimensional Bratu-type problem, gets its significance from the first thermal combustion theory, which was developed by simplifying the solid fuel ignition model. To this end, so many works have been done on Bratu-type differential equation. Salema and Thanoona (2022) used perturbation Method for the Solution of Bratu's type equation. Zarebnia and Hoshyar (2014) used spline method for the solution of Bratu's type equation. Wazwaz (2016) used the method of successive differentiation for solving Bratu's type equations. Saravi et al (2013) used the He's variational Iteration Method for the solution of Bratu's equation. Fenta and Derese (2019) used sixth-order Runge-Kutta seven stages method for the Numerical solution of second order initial value problems of Bratu-type equations. Ezekiel (2013) used New Improved Variational Homotopy Perturbation Method for Bratu-Type Problems. Hassan and Semary (2013) used Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method, while Oyedepo et al. (2023) applied Homotopy Perturbation Technique on fractional Volterra and Fredholm Integro integro-differential equations. Also, Otaide & Oluwayemi (2024) considered a numerical treatment of linear Volterra Integro differential equations using variational iteration algorithm with collocation. Aregbesola (2003) used the method of weighted residual for the Numerical solution of Bratu problem. Wazwaz (2005) used Adomian's decomposition method for the treatment of Bratu's type equations. Changqing and Jianhua (2013) used the Chebyshev wavelets method for the solution of Bratu's problem. In this work, the Taylor series technique is used to generate the Numerical solution of the Bratu's nonlinear differential equation (1) with given initial conditions. The proposed method works efficiently and the results so far are very encouraging and reliable. Finally, the Numerical solution is given in an infinite series and converges absolutely.

Materials and Methods

Taylor's Series Technique: The Taylor series o a real-valued function $f(x)$ that is infinitely differentiable at $x = x_0 \in \Re$ is given by

$$w(x) = \sum_{n=0}^{\infty} \frac{w^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$w(x) = w(x_0) + \frac{w'(x_0)}{1!}(x - x_0) + \frac{w''(x_0)}{2!}(x - x_0)^2 + \frac{w'''(x_0)}{3!}(x - x_0)^3 + \cdots \qquad (3)$$

From (1), we have the following:

$$w(x) = -\gamma \int_0^x \left(\int_0^s e^{\alpha\, w(t)} dt \right) ds$$

$$\frac{d}{dx} w(x) = -\gamma \left(\int_0^x e^{\alpha\, w(s)}\, ds \right)$$

$$\frac{d^2}{dx^2} w(x) = -\gamma e^{\alpha\, w(x)}$$

$$\frac{d^3}{dx^3} w(x) = -\gamma\alpha \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)}$$

$$\frac{d^4}{dx^4} w(x) = -\gamma\alpha \left(\frac{d^2}{dx^2} w(x) \right) e^{\alpha\, w(x)} - \gamma\alpha^2 \left(\frac{d}{dx} w(x) \right)^2 e^{\alpha\, w(x)}$$

$$\begin{aligned}\frac{d^5}{dx^5} w(x) = {} & -\gamma\alpha \left(\frac{d^3}{dx^3} w(x) \right) e^{\alpha\, w(x)} - 3\gamma\alpha^2 \left(\frac{d^2}{dx^2} w(x) \right) \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} \\ & - \gamma\alpha^3 \left(\frac{d}{dx} w(x) \right)^3 e^{\alpha\, w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^6}{dx^6} w(x) = {} & -\gamma\alpha \left(\frac{d^4}{dx^4} w(x) \right) e^{\alpha\, w(x)} - 4\gamma\alpha^2 \left(\frac{d^3}{dx^3} w(x) \right) \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} \\ & - 3\gamma\alpha^2 \left(\frac{d^2}{dx^2} w(x) \right)^2 e^{\alpha\, w(x)} - 6\gamma\alpha^3 \left(\frac{d^2}{dx^2} w(x) \right) \left(\frac{d}{dx} w(x) \right)^2 e^{\alpha\, w(x)} \\ & - \gamma\alpha^4 \left(\frac{d}{dx} w(x) \right)^4 e^{\alpha\, w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^7}{dx^7} w(x) = {} & -\gamma\alpha \left(\frac{d^5}{dx^5} w(x) \right) e^{\alpha\, w(x)} - 5\gamma\alpha^2 \left(\frac{d^4}{dx^4} w(x) \right) \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} \\ & - 10\gamma\alpha^2 \left(\frac{d^3}{dx^3} w(x) \right) \left(\frac{d^2}{dx^2} w(x) \right) e^{\alpha\, w(x)} - 10\gamma\alpha^3 \left(\frac{d^3}{dx^3} w(x) \right) \left(\frac{d}{dx} w(x) \right)^2 e^{\alpha\, w(x)} \\ & - 15\gamma\alpha^3 \left(\frac{d^2}{dx^2} w(x) \right)^2 \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} - 10\gamma\alpha^4 \left(\frac{d^2}{dx^2} w(x) \right) \left(\frac{d}{dx} w(x) \right)^3 e^{\alpha\, w(x)} \\ & - \gamma\alpha^5 \left(\frac{d}{dx} w(x) \right)^5 e^{\alpha\, w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^8}{dx^8} w(x) = {} & -\gamma\alpha \left(\frac{d^6}{dx^6} w(x) \right) e^{\alpha\, w(x)} - 6\gamma\alpha^2 \left(\frac{d^5}{dx^5} w(x) \right) \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} \\ & - 15\gamma\alpha^2 \left(\frac{d^4}{dx^4} w(x) \right) \left(\frac{d^2}{dx^2} w(x) \right) e^{\alpha\, w(x)} - 15\gamma\alpha^3 \left(\frac{d^4}{dx^4} w(x) \right) \left(\frac{d}{dx} w(x) \right)^2 e^{\alpha\, w(x)} \\ & - 10\gamma\alpha^2 \left(\frac{d^3}{dx^3} w(x) \right)^2 e^{\alpha\, w(x)} - 60\gamma\alpha^3 \left(\frac{d^3}{dx^3} w(x) \right) \left(\frac{d^2}{dx^2} w(x) \right) \left(\frac{d}{dx} w(x) \right) e^{\alpha\, w(x)} \\ & - 20\gamma\alpha^4 \left(\frac{d^3}{dx^3} w(x) \right) \left(\frac{d}{dx} w(x) \right)^3 e^{\alpha\, w(x)} - 15\gamma\alpha^3 \left(\frac{d^2}{dx^2} w(x) \right)^3 e^{\alpha\, w(x)} \\ & - 45\gamma\alpha^4 \left(\frac{d^2}{dx^2} w(x) \right)^2 \left(\frac{d}{dx} w(x) \right)^2 e^{\alpha\, w(x)} \\ & - 15\gamma\alpha^5 \left(\frac{d^2}{dx^2} w(x) \right) \left(\frac{d}{dx} w(x) \right)^4 e^{\alpha\, w(x)} - \gamma\alpha^6 \left(\frac{d}{dx} w(x) \right)^6 e^{\alpha\, w(x)}\end{aligned}$$

Setting $x = x_0 = 0$; from (2), we obtain:

$$w(x) = w(0) + \frac{w'(0)}{1!}x + \frac{w''(0)}{2!}x^2 + \frac{w'''(0)}{3!}x^3 + \cdots \qquad (4)$$

where (4) is called the Maclaurin's Series.

Numerical Applications

In this section, we solved two Bratu-type problems using the suggested approach. The correctness, effectiveness, and dependability of the suggested approach are also demonstrated by the numerical outcomes.

Problem 1: Salema and Thanoona (2022)

Consider the Bratu-type problem (1), by choosing $\gamma = -2$ and $\alpha = 1$ to obtain:

$$\frac{d^2w}{dx^2} - 2e^w = 0, \quad 0 < x < 1, \qquad (5)$$

$$w(0) = w'(0) = 0 \qquad (6)$$

The exact solution of the system Salema & Thanoona (2022), is given as:

$$w(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \cdots.$$

Applying the Taylor's Series technique, we have the following:

$$w(x) = 2\int_0^x \left(\int_0^s e^{w(t)} dt \right) ds$$

$$\frac{d}{dx} w(x) = 2\left(\int_0^x e^{w(s)}\, ds \right)$$

$$\frac{d^2}{dx^2} w(x) = 2\, e^{w(x)}$$

$$\frac{d^3}{dx^3} w(x) = 2\left(\frac{d}{dx} w(x) \right) e^{w(x)}$$

$$\frac{d^4}{dx^4} w(x) = 2\left(\frac{d^2}{dx^2} w(x) \right) e^{w(x)} + 2\left(\frac{d}{dx} w(x) \right)^2 e^{w(x)}$$

$$\frac{d^5}{dx^5} w(x) = 2\left(\frac{d^3}{dx^3} w(x) \right) e^{w(x)} + 6\left(\frac{d^2}{dx^2} w(x) \right)\left(\frac{d}{dx} w(x) \right) e^{w(x)} + 2\left(\frac{d}{dx} w(x) \right)^3 e^{w(x)}$$

$$\frac{d^6}{dx^6} w(x) = 2\left(\frac{d^4}{dx^4} w(x) \right) e^{w(x)} + 8\left(\frac{d^3}{dx^3} w(x) \right)\left(\frac{d}{dx} w(x) \right) e^{w(x)}$$
$$+ 6\left(\frac{d^2}{dx^2} w(x) \right)^2 e^{w(x)} + 12\left(\frac{d^2}{dx^2} w(x) \right)\left(\frac{d}{dx} w(x) \right)^2 e^{w(x)} + 2\left(\frac{d}{dx} w(x) \right)^4 e^{w(x)}$$

$$\frac{d^7}{dx^7} w(x) = 2\left(\frac{d^5}{dx^5} w(x)\right) e^{w(x)} + 10\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{w(x)} + 20\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right) e^{w(x)} + 20\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d}{dx} w(x)\right)^2 e^{w(x)} + 30\left(\frac{d^2}{dx^2} w(x)\right)^2\left(\frac{d}{dx} w(x)\right) e^{w(x)} + 20\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right)^3 e^{w(x)} + 2\left(\frac{d}{dx} w(x)\right)^5 e^{w(x)}$$

$$\frac{d^8}{dx^8} w(x) = 2\left(\frac{d^6}{dx^6} w(x)\right) e^{w(x)} + 12\left(\frac{d^5}{dx^5} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{w(x)} + 30\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right) e^{w(x)} + 30\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d}{dx} w(x)\right)^2 e^{w(x)} + 20\left(\frac{d^3}{dx^3} w(x)\right)^2 e^{w(x)} + 120\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{w(x)} + 40\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d}{dx} w(x)\right)^3 e^{w(x)} + 30\left(\frac{d^2}{dx^2} w(x)\right)^3 e^{w(x)} + 90\left(\frac{d^2}{dx^2} w(x)\right)^2\left(\frac{d}{dx} w(x)\right)^2 e^{w(x)} + 30\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right)^4 e^{w(x)} + 2\left(\frac{d}{dx} w(x)\right)^6 e^{w(x)}$$

Using (3) we have that:

$$w(x) = w(x_0) + \frac{w'(x_0)}{1!}(x - x_0) + \frac{w''(x_0)}{2!}(x - x_0)^2 + \frac{w'''(x_0)}{3!}(x - x_0)^3 + \frac{w^{(iv)}(x_0)}{4!}(x - x_0)^4 + \frac{w^{(v)}(x_0)}{5!}(x - x_0)^5 + \frac{w^{(vi)}(x_0)}{6!}(x - x_0)^6 + \frac{w^{(vii)}(x_0)}{7!}(x - x_0)^7 + \frac{w^{(viii)}(x_0)}{8!}(x - x_0)^8 + \cdots.$$

Applying (6), the Series solution is given as

$$w(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \cdots.$$

Problem 2: Salema and Thanoona (2022)

Consider the Bratu-type problem (1), by choosing $\gamma = -1$ and $\alpha = 2$ to obtain:

$$\frac{d^2w}{dx^2} - e^{2w} = 0, \quad 0 < x < 1, \qquad (7)$$

$$w(0) = w'(0) = 0 \qquad (8)$$

The exact solution of the system Salema and Thanoona (2022), is given as:

$$w(x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \cdots$$

Applying the Taylor's Series technique, we have the following:

$$w(x) = \int_0^x \left(\int_0^s e^{2w(t)} dt\right) ds$$

Cite this article as:
Otaide, I.J., & Ugbene I.J. (2025). Application of the Taylor series technique to the solution of Bratu problems. *FNAS Journal of Mathematical Modeling and Numerical Simulation*, *2*(2), 19-26.

$$\frac{d}{dx} w(x) = \int_0^x e^{2\,w(s)}\, ds$$

$$\frac{d^2}{dx^2} w(x) = e^{2\,w(x)}$$

$$\frac{d^3}{dx^3} w(x) = 2\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)}$$

$$\frac{d^4}{dx^4} w(x) = 2\left(\frac{d^2}{dx^2} w(x)\right) e^{2\,w(x)} + 4\left(\frac{d}{dx} w(x)\right)^2 e^{2\,w(x)}$$

$$\begin{aligned}\frac{d^5}{dx^5} w(x) &= 2\left(\frac{d^3}{dx^3} w(x)\right) e^{2\,w(x)} + 12\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} \\ &\quad + 8\left(\frac{d}{dx} w(x)\right)^3 e^{2\,w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^6}{dx^6} w(x) &= 2\left(\frac{d^4}{dx^4} w(x)\right) e^{2\,w(x)} + 16\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} \\ &\quad + 12\left(\frac{d^2}{dx^2} w(x)\right)^2 e^{2\,w(x)} + 48\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right)^2 e^{2\,w(x)} \\ &\quad + 16\left(\frac{d}{dx} w(x)\right)^4 e^{2\,w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^7}{dx^7} w(x) &= 2\left(\frac{d^5}{dx^5} w(x)\right) e^{2\,w(x)} + 20\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} \\ &\quad + 40\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right) e^{2\,w(x)} + 80\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d}{dx} w(x)\right)^2 e^{2\,w(x)} \\ &\quad + 120\left(\frac{d^2}{dx^2} w(x)\right)^2\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} + 160\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right)^3 e^{2\,w(x)} \\ &\quad + 32\left(\frac{d}{dx} w(x)\right)^5 e^{2\,w(x)}\end{aligned}$$

$$\begin{aligned}\frac{d^8}{dx^8} w(x) &= 2\left(\frac{d^6}{dx^6} w(x)\right) e^{2\,w(x)} + 24\left(\frac{d^5}{dx^5} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} \\ &\quad + 60\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right) e^{2\,w(x)} + 120\left(\frac{d^4}{dx^4} w(x)\right)\left(\frac{d}{dx} w(x)\right)^2 e^{2\,w(x)} \\ &\quad + 40\left(\frac{d^3}{dx^3} w(x)\right)^2 e^{2\,w(x)} + 480\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right) e^{2\,w(x)} \\ &\quad + 320\left(\frac{d^3}{dx^3} w(x)\right)\left(\frac{d}{dx} w(x)\right)^3 e^{2\,w(x)} + 120\left(\frac{d^2}{dx^2} w(x)\right)^3 e^{2\,w(x)} \\ &\quad + 720\left(\frac{d^2}{dx^2} w(x)\right)^2\left(\frac{d}{dx} w(x)\right)^2 e^{2\,w(x)} + 480\left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d}{dx} w(x)\right)^4 e^{2\,w(x)} \\ &\quad + 64\left(\frac{d}{dx} w(x)\right)^6 e^{2\,w(x)}\end{aligned}$$

Using (3) we have that:

$$w(x) = w(x_0) + \frac{w'(x_0)}{1!}(x - x_0) + \frac{w''(x_0)}{2!}(x - x_0)^2 + \frac{w'''(x_0)}{3!}(x - x_0)^3 + \frac{w^{(iv)}(x_0)}{4!}(x - x_0)^4 + \frac{w^{(v)}(x_0)}{5!}(x - x_0)^5 + \frac{w^{(vi)}(x_0)}{6!}(x - x_0)^6 + \frac{w^{(vii)}(x_0)}{7!}(x - x_0)^7 + \frac{w^{(viii)}(x_0)}{8!}(x - x_0)^8 + \cdots$$

Applying (8), the Series solution is given as

$$w(x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \cdots$$

Results

Table 1: Comparison of Salema & Thanoona (2022) Perturbation Method and the suggested approach for problem 1.

x	Precise solution	Approximate solution	Absolute Error by the suggested technique	Absolute error by Perturbation Method
0.1	0.01001671124	0.01001671124	0.00000000e-00	1.117×10^{-8}
0.2	0.04026954565	0.04026954565	0.00000000e-00	7.2635×10^{-7}
0.3	0.09138328521	0.09138328521	0.00000000e-00	0.00000849054
0.4	0.1644575533	0.1644575533	0.00000000e-00	0.0000494120
0.5	0.2611638145	0.2611638145	0.00000000e-00	0.0001968626
0.6	0.3839002149	0.3839002149	0.00000000e-00	0.0006183771
0.7	0.5360233017	0.5360233017	0.00000000e-00	0.0016503540
0.8	0.7221811037	0.7221811037	0.00000000e-00	0.0039113549
0.9	0.9487774909	0.9487774909	0.00000000e-00	0.0084672048

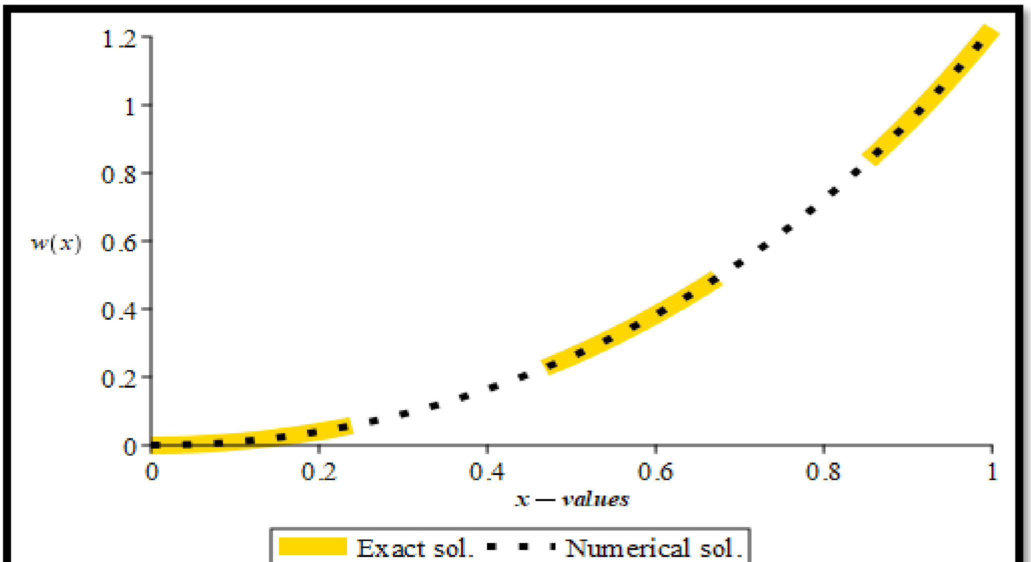

figure 1: Graphical comparison of approximate and exact solution for problem 1

Table 2: Comparison of Salema & Thanoona (2022) Perturbation Method and the suggested approach for problem 2.

x	Precise solution	Approximate solution	Absolute Error by the suggested technique	Absolute error by Perturbation Method
0.1	0.005008355622	0.005008355622	0.00000000e-00	2.3386×10^{-8}
0.2	0.02013477282	0.02013477282	0.00000000e-00	0.00000149142
0.3	0.04569164261	0.04569164261	0.00000000e-00	0.00001688860
0.4	0.08222877663	0.08222877663	0.00000000e-00	0.00009410781
0.5	0.1305819072	0.1305819072	0.00000000e-00	0.0003551506
0.6	0.1919501074	0.1919501074	0.00000000e-00	0.0010464518
0.7	0.2680116508	0.2680116508	0.00000000e-00	0.0025969723
0.8	0.3610905518	0.3610905518	0.00000000e-00	0.0056791045
0.9	0.4743887455	0.4743887455	0.00000000e-00	0.0112663280

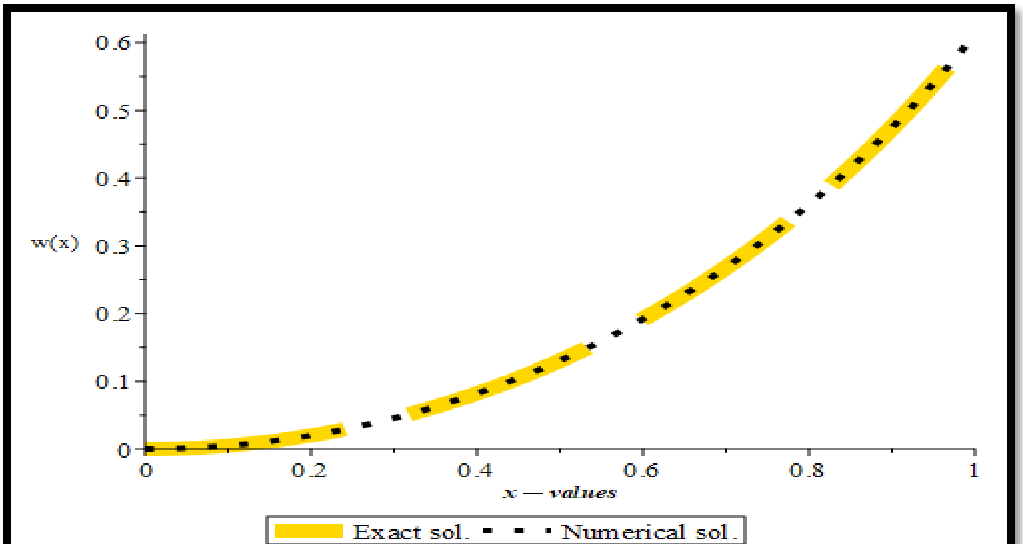

figure 2: Graphical comparison of approximate and exact solution for problem 2

Discussion

The study's significance lies in its capacity to use the Taylor series approach to solve non-linear Bratu differential equations in an accurate and efficient manner. Applications for the work can be found in mathematics as well as a number of other scientific and engineering disciplines. This work is significant because it advances mathematical techniques for resolving Bratu differential equations. As can be seen from problems 1 and 2, there is an absolute convergence between the approximate and exact solutions. In broad fields of study, the study generally has both theoretical and practical applications.

Conclusion

In order to acquire numerical solutions for Bratu-type equations, the Taylor's Series technique has been successfully employed in this research. The approach provides series solutions that are totally convergent and arise in physical problems. Tables 1 and 2 show that, in comparison to methods found in the literature, the suggested approach produces a better result. Finally, the numerical results showed that the existing method offers a powerful mathematical tool for resolving the class of problem under consideration.

Recommendation

The Bratu-type equations are crucial to engineering and research. Therefore, the future direction of our work is to verify the efficiency and convergence of the solution by using the same numerical technique to solve boundary value problems or, most likely, integral and integro differential equations.

References

Aregbesola, Y. (2003). Numerical solution of Bratu problem using the method of weighted residual, Elect. J. South African Math. Soc., **3**(1), 1–7.

Changqing, Y., & Jianhua H. (2013). Chebyshev wavelets method for solving Bratu's problem, Bound. Value Prob.(1), 1–9.

Ezekiel A. (2013). New Improved Variational Homotopy Perturbation Method for Bratu-Type Problems, *Amer. J. Comput. Math*., **3**(2),110–113.

Fenta, H. B., & Derese G. A. (2019). Numerical solution of second order initial value problems of Bratu-type equations using sixth order Runge-Kutta seven stages method, *Int. J. Comput. Sci. Appl. Math*., **5**(1),

Hassan, H. N., & Semary, M. S. (2013). Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method, *Commun. Numerical Anal*, 1-14.

Otaide I. J., & Oluwayemi M. O. (2024). Numerical treatment of linear volterra integro differential equations using variational iteration algorithm with collocation, *Partial Differential Equations and Applied Mathematics*, doi: https://doi.org/10.1016/j.padiff.2024.100693.

Oyedepo, T., Oluwayemi, M. O., Ayinde, M. A., Osilagun, J. A., &Ahmed L. O. (2023). Homotopy Perturbation Technique for Fractional Volterra and Fredholm Integro Differential Equations,"*International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG)*, Omu-Aran, Nigeria,1-6, https://doi.org/10.1109/SEB-SDG57117.2023.10124633.

Salem, S. A., & Thanoon, T. Y. (2022). On solving Bratu's type equation by perturbation method. *International Journal of Nonlinear Analysis and Applications*, *13*(1), 2755-2763.

Saravi, M., Hermann, M., & Kaiser, D. (2013). Solution of Bratu's Equation by He's variational Iteration Method, *Amer. J. Comput. Appl. Math*, **3**(1), 46–48.

Wazwaz, A. M. (2005). Adomians decomposition method for a reliable treatment of the Bratu-type equations, *Appl. Math. Comput.*, **166**, 652–663.

Wazwaz, A. M. (2016). The successive differentiation method for solving Bratu equation and Bratu-type equations, *Rom. J. Phys.*, **61**, 774–783.

Zarebnia M., & Hoshyar M. (2014). Solution of Bratu-type equation via spline method, *Acta Universities Apulensis*, **37**, 61–72.