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Application of Linear Block Methods for Solving Higher-Order Initial Value Problems

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Abstract
This research investigates the challenges of solving higher-order oscillatory ordinary differential equations (ODEs), which frequently arise in various scientific and engineering applications. Many of these problems lack explicit solutions, necessitating the development of robust numerical methods. This study proposes a novel approach to solving such equations by employing a two-step linear multistep method specifically designed for directly addressing higher-order oscillatory differential equations. Key numerical properties, including consistency, zero stability, convergence, and linear stability, are thoroughly analyzed to validate the effectiveness of the proposed scheme. The effectiveness of the new method is illustrated through a series of numerical examples, highlighting its accuracy and efficiency compared to existing methods in the literature. The results demonstrate that the proposed method outperforms traditional techniques in solving complex oscillatory problems, providing reliable and computationally efficient solutions suitable for real-world applications.

Keywords: Consistency, Convergence, Computational Efficiency, Higher-Order Oscillatory, Linear Block Approach, Numerical Methods

Introduction

Many physical problems in science, social science and technology remain unexplored or insufficiently addressed, despite some receiving research attention. Oscillatory phenomena, critical in these fields, are effectively modeled by differential equations, emphasizing the need for further study and innovation (Blanka, 2019; Adewale & Sabo, 2024; Sabo et al., 2024). Since explicit solutions for many higher-order ordinary differential equations (ODEs) are unavailable, developing numerical methods like implicit linear multistep methods (LMMs) is essential. This research focuses on solving initial value problems for general higher-order ODEs, addressing the challenges posed by their complexity.

$$y^{(\lambda)}(t) = f\left(t, y, y^{(1)}, \cdots, y^{(\lambda-1)}\right), \; y(a_0) = \tau_0, \; y'(a_1) = \tau_1, \cdots, \; y^{(\lambda-1)}(a_\mu) = \tau_\mu \qquad (1)$$

Higher-order derivatives are traditionally solved using predictor-corrector methods, where predictors aid in implementing the corrector, and Taylor series expansions p are utilized to establish initial values. Research suggests that direct methods often yield more accurate and convenient results compared to reducing systems to first-order ordinary differential equations (Ukpebor et al., 2020). Notably, Ayinde et al. (2023) have developed schemes for solving second-order oscillatory differential equations modeling dynamic motion. Similarly, approaches for third-order differential equations have been proposed by Duromola (2022), Kashkari and Alqarni (2019), Folaranmi et al. (2021), and Sabo et al. (2022). Additionally, researchers such as Adeyeye, and Omar, (2019), Modebei et al. (2019), and Raymond et al. (2023) have applied their methods directly to these complex problems. Researchers such as Adeyefa and Kuboye (2020), Donald et al. (2024), Skwame et al. (2024), Elnady et al. (2024), and Workineh et al. (2024) have focused on solving second, third, and fourth-order initial value problems using advanced numerical methods. By directly addressing these higher-order equations, their work aims to improve accuracy and efficiency in both practical and theoretical applications across fields like engineering, physics, and applied mathematics. These studies often develop or refine methods tailored to handle the inherent complexities of higher-order differential equations, offering reliable and computationally efficient solutions. Their contributions highlight the growing importance of direct approaches in solving complex systems, eliminating the need for reduction to first-order systems and advancing numerical analysis as a discipline (Adeyefa, & Kuboye, 2020; Abolarin et al., 2020; Donald et al., 2024).

Higher-order oscillatory differential equations, often arising in simulations of physical phenomena, pose significant challenges due to the limited availability of analytical solutions. Traditional methods frequently involve transforming these equations into first-order systems, which can complicate the solution process and increase computational demands. Addressing these issues, researchers like Abdulrahim (2021) have proposed innovative numerical techniques for directly solving higher-order initial value problems. Their work focuses on enhancing the accuracy and efficiency of these methods, providing robust tools for tackling complex differential equations in theoretical and applied contexts. These efforts represent significant advancements in both the theory and application of numerical methods for higher-order ordinary differential equations.

Construction of Linear Block Approach

The two-step linear multistep method

$$\sum_{i=0}^{2}\alpha_i y_{n+i} = h^4 \sum_{i=0}^{2}\beta_i f_{n+i} \qquad (2)$$

is consider. Derivation of a new method for the direct solution of higher-order oscillatory differential equations (1) using the linear block approach was done.

where $Y_{n+k} = \left(y_{n+a},\, y_{n+b}, \cdots,\, y_{n+k}\right)$ and $Y^{(i)}_{n+k} = \left(y^{(i)}_{n+a},\, y^{(i)}_{n+b}, \cdots,\, y^{(i)}_{n+k}\right)$. To determine the unknown values, the generalized algorithm is applied.

$$y_{n+\tau} = \sum_{i=0}^{3}\frac{(\tau h)^j}{i!}y_n^{(j)} + \sum_{i=0}^{2}\left(\omega_{i\xi} f_{n+i}\right), \tau = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \qquad (3)$$

its fourth derivatives are

$$y^{iv}_{n+\tau} = \sum_{i=0}^{4-(\tau+1)}\frac{(\tau h)^i}{i!}y_n^{(i+\tau)}, \tau = 1_{\left(\tau=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2k\right)}, \tau = 2_{\left(\tau=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2k\right)}, \tau = 3_{\left(\tau=0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2k\right)} \qquad (4)$$

is consider, with $\omega_{\tau i} = \mathrm{X}^{-1}\mathrm{M}$ and $\Omega_{\tau i} = \mathrm{X}^{-1}E$ where

$$\mathrm{X} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{h}{8} & \frac{h}{4} & \frac{3h}{4} & \frac{h}{2} & \frac{5h}{4} & \frac{3h}{2} & \frac{7h}{4} & 2h \\ 0 & \frac{\left(\frac{h}{4}\right)^2}{2!} & \frac{\left(\frac{h}{2}\right)^2}{2!} & \frac{\left(\frac{3h}{4}\right)^2}{2!} & \frac{(h)^2}{2!} & \frac{\left(\frac{5h}{4}\right)^2}{2!} & \frac{\left(\frac{3h}{2}\right)^2}{2!} & \frac{\left(\frac{7h}{4}\right)^2}{2!} & \frac{(2h)^2}{2!} \\ 0 & \frac{\left(\frac{h}{4}\right)^3}{3!} & \frac{\left(\frac{h}{2}\right)^3}{3!} & \frac{\left(\frac{3h}{4}\right)^3}{3!} & \frac{(h)^3}{3!} & \frac{\left(\frac{5h}{4}\right)^3}{3!} & \frac{\left(\frac{3h}{2}\right)^3}{3!} & \frac{\left(\frac{7h}{4}\right)^3}{3!} & \frac{(2h)^3}{3!} \\ 0 & \frac{\left(\frac{h}{4}\right)^4}{4!} & \frac{\left(\frac{h}{2}\right)^4}{4!} & \frac{\left(\frac{3h}{4}\right)^4}{4!} & \frac{(h)^4}{4!} & \frac{\left(\frac{5h}{4}\right)^4}{4!} & \frac{\left(\frac{3h}{2}\right)^4}{4!} & \frac{\left(\frac{7h}{4}\right)^4}{4!} & \frac{(2h)^4}{4!} \\ 0 & \frac{\left(\frac{h}{4}\right)^5}{5!} & \frac{\left(\frac{h}{2}\right)^5}{5!} & \frac{\left(\frac{3h}{4}\right)^5}{5!} & \frac{(h)^5}{5!} & \frac{\left(\frac{5h}{4}\right)^5}{5!} & \frac{\left(\frac{3h}{2}\right)^5}{5!} & \frac{\left(\frac{7h}{4}\right)^5}{5!} & \frac{(2h)^5}{5!} \\ 0 & \frac{\left(\frac{h}{4}\right)^6}{6!} & \frac{\left(\frac{h}{2}\right)^6}{6!} & \frac{\left(\frac{3h}{4}\right)^6}{6!} & \frac{(h)^6}{6!} & \frac{\left(\frac{5h}{4}\right)^6}{6!} & \frac{\left(\frac{3h}{2}\right)^6}{6!} & \frac{\left(\frac{7h}{4}\right)^6}{6!} & \frac{(2h)^6}{6!} \\ 0 & \frac{\left(\frac{h}{4}\right)^7}{7!} & \frac{\left(\frac{h}{2}\right)^7}{7!} & \frac{\left(\frac{3h}{4}\right)^7}{7!} & \frac{(h)^7}{7!} & \frac{\left(\frac{5h}{4}\right)^7}{7!} & \frac{\left(\frac{3h}{2}\right)^7}{7!} & \frac{\left(\frac{7h}{4}\right)^7}{7!} & \frac{(2h)^7}{7!} \\ 0 & \frac{\left(\frac{h}{4}\right)^8}{8!} & \frac{\left(\frac{h}{2}\right)^8}{8!} & \frac{\left(\frac{3h}{4}\right)^8}{8!} & \frac{(h)^8}{8!} & \frac{\left(\frac{5h}{4}\right)^8}{8!} & \frac{\left(\frac{3h}{2}\right)^8}{8!} & \frac{\left(\frac{7h}{4}\right)^8}{8!} & \frac{(2h)^8}{8!} \end{pmatrix}, \quad \mathrm{M} = \begin{pmatrix} \frac{(\tau h)^4}{4!} \\ \frac{(\tau h)^5}{5!} \\ \frac{(\tau h)^6}{6!} \\ \frac{(\tau h)^7}{7!} \\ \frac{(\tau h)^8}{8!} \\ \frac{(\tau h)^9}{9!} \\ \frac{(\tau h)^{10}}{10!} \\ \frac{(\tau h)^{11}}{11!} \\ \frac{(\tau h)^{12}}{12!} \end{pmatrix}, \quad \mathrm{E} = \begin{pmatrix} \frac{(\tau h)^{4-\zeta}}{(4-\zeta)!} \\ \frac{(\tau h)^{5-\zeta}}{(5-\zeta)!} \\ \frac{(\tau h)^{6-\zeta}}{(6-\zeta)!} \\ \frac{(\tau h)^{7-\zeta}}{(7-\zeta)!} \\ \frac{(\tau h)^{8-\zeta}}{(8-\zeta)!} \\ \frac{(\tau h)^{9-\zeta}}{(9-\zeta)!} \\ \frac{(\tau h)^{10-\zeta}}{(10-\zeta)!} \\ \frac{(\tau h)^{11-\zeta}}{(11-\zeta)!} \\ \frac{(\tau h)^{12-\zeta}}{(12-\zeta)!} \end{pmatrix}$$

So, to derive the new methods, the subsequent Preposition were proved.

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Preposition 1

The linear multistep method (2), when combined with the linear block approach described in (3) and (4), exclusively utilizes a block method structure. We then generalize this corollary to construct higher-order schemes derived from the block algorithm.

This can be verified with the help of the equation (3) and (4) as a block at the points $\left(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right)$.

Proof

Now, by simplifying (3) and (4) using the partitioned points and solving these equations sequentially, the coefficients of the polynomial are determined. $y_{n\tau}, \tau = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$

Substituting $\tau = \xi_n + xh$, the polynomial takes the form

$$y(\tau_n + xh) = \alpha_{\frac{1}{4}} y_{n+\frac{1}{4}} + \alpha_1 y_{n+1} + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + \alpha_2 y_{n+2} + h^4 \begin{pmatrix} \beta_0 f_n + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} \\ + \beta_{\frac{5}{4}} f_{n+\frac{5}{4}} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_{\frac{7}{4}} f_{n+\frac{7}{4}} + \beta_2 f_{n+2} \end{pmatrix} \tag{5}$$

The block algorithm (3) is expanded to yield

$$\left.\begin{array}{l}
y_{n+\frac{1}{4}} = y_n + \frac{1}{4} h y'_n + \frac{\left(\frac{1}{4}h\right)^2}{2!} y''_n + \frac{\left(\frac{1}{4}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{011} f_n + \omega_{012} f_{n+\frac{1}{4}} + \omega_{013} f_{n+\frac{1}{2}} + \omega_{014} f_{n+\frac{3}{4}} + \omega_{015} f_{n+1} \\ + \omega_{016} f_{n+\frac{5}{4}} + \omega_{017} f_{n+\frac{3}{2}} + \omega_{018} f_{n+\frac{7}{4}} + \omega_{019} f_{n+1} \end{pmatrix} \\
y_{n+\frac{1}{2}} = y_n + \frac{1}{2} h y'_n + \frac{\left(\frac{1}{2}h\right)^2}{2!} y''_n + \frac{\left(\frac{1}{2}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{021} f_n + \omega_{022} f_{n+\frac{1}{4}} + \omega_{023} f_{n+\frac{1}{2}} + \omega_{024} f_{n+\frac{3}{4}} + \omega_{025} f_{n+1} \\ + \omega_{026} f_{n+\frac{5}{4}} + \omega_{027} f_{n+\frac{3}{2}} + \omega_{028} f_{n+\frac{7}{4}} + \omega_{029} f_{n+1} \end{pmatrix} \\
y_{n+\frac{3}{4}} = y_n + \frac{3}{4} h y'_n + \frac{\left(\frac{3}{4}h\right)^2}{2!} y''_n + \frac{\left(\frac{3}{4}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{031} f_n + \omega_{032} f_{n+\frac{1}{4}} + \omega_{033} f_{n+\frac{1}{2}} + \omega_{034} f_{n+\frac{3}{4}} + \omega_{035} f_{n+1} \\ + \omega_{036} f_{n+\frac{5}{4}} + \omega_{037} f_{n+\frac{3}{2}} + \omega_{038} f_{n+\frac{7}{4}} + \omega_{039} f_{n+1} \end{pmatrix} \\
y_{n+1} = y_n + h y'_n + \frac{(h)^2}{2!} y''_n + \frac{(h)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{041} f_n + \omega_{042} f_{n+\frac{1}{4}} + \omega_{043} f_{n+\frac{1}{2}} + \omega_{044} f_{n+\frac{3}{4}} + \omega_{045} f_{n+1} \\ + \omega_{046} f_{n+\frac{5}{4}} + \omega_{047} f_{n+\frac{3}{2}} + \omega_{048} f_{n+\frac{7}{4}} + \omega_{049} f_{n+1} \end{pmatrix} \\
y_{n+\frac{5}{4}} = y_n + \frac{5}{4} h y'_n + \frac{\left(\frac{5}{4}h\right)^2}{2!} y''_n + \frac{\left(\frac{5}{4}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{051} f_n + \omega_{052} f_{n+\frac{1}{4}} + \omega_{053} f_{n+\frac{1}{2}} + \omega_{054} f_{n+\frac{3}{4}} + \omega_{055} f_{n+1} \\ + \omega_{056} f_{n+\frac{5}{4}} + \omega_{057} f_{n+\frac{3}{2}} + \omega_{058} f_{n+\frac{7}{4}} + \omega_{059} f_{n+1} \end{pmatrix} \\
y_{n+\frac{3}{2}} = y_n + \frac{3}{2} h y'_n + \frac{\left(\frac{3}{2}h\right)^2}{2!} y''_n + \frac{\left(\frac{3}{2}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{061} f_n + \omega_{062} f_{n+\frac{1}{4}} + \omega_{063} f_{n+\frac{1}{2}} + \omega_{064} f_{n+\frac{3}{4}} + \omega_{065} f_{n+1} \\ + \omega_{066} f_{n+\frac{5}{4}} + \omega_{067} f_{n+\frac{3}{2}} + \omega_{068} f_{n+\frac{7}{4}} + \omega_{069} f_{n+1} \end{pmatrix} \\
y_{n+\frac{7}{4}} = y_n + \frac{7}{4} h y'_n + \frac{\left(\frac{7}{4}h\right)^2}{2!} y''_n + \frac{\left(\frac{7}{4}h\right)^3}{3!} y'''_n + h^4 \begin{pmatrix} \omega_{071} f_n + \omega_{072} f_{n+\frac{1}{4}} + \omega_{073} f_{n+\frac{1}{2}} + \omega_{074} f_{n+\frac{3}{4}} + \omega_{075} f_{n+1} \\ + \omega_{076} f_{n+\frac{5}{4}} + \omega_{077} f_{n+\frac{3}{2}} + \omega_{078} f_{n+\frac{7}{4}} + \omega_{079} f_{n+1} \end{pmatrix} \\
y_{n+2} = y_n + 2h y'_n + \frac{(2h)^2}{2!} y''_n + \frac{(2h)^3}{3!} y'''_n + h^4 \omega \begin{pmatrix} \psi_{081} f_n + \omega_{082} f_{n+\frac{1}{4}} + \omega_{083} f_{n+\frac{1}{2}} + \omega_{084} f_{n+\frac{3}{4}} + \psi_{085} f_{n+1} \\ + \psi_{086} f_{n+\frac{5}{4}} + \psi_{087} f_{n+\frac{3}{2}} + \psi_{088} f_{n+\frac{7}{4}} + \psi_{089} f_{n+1} \end{pmatrix}
\end{array}\right\} \tag{6}$$

Similarly, the linear block algorithm (6) is extended to derive the higher-order derivatives as

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$$\left.\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + \frac{\left(\frac{1}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{111}f_n + \Omega_{112}f_{n+\frac{1}{4}} + \Omega_{113}f_{n+\frac{1}{2}} + \Omega_{114}f_{n+\frac{3}{4}} + \Omega_{115}f_{n+1} + \Omega_{116}f_{n+\frac{5}{4}} + \Omega_{117}f_{n+\frac{3}{2}} + \Omega_{118}f_{n+\frac{7}{4}} + \Omega_{119}f_{n+2}\right)\\
y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{\left(\frac{1}{2}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{121}f_n + \Omega_{122}f_{n+\frac{1}{4}} + \Omega_{123}f_{n+\frac{1}{2}} + \Omega_{124}f_{n+\frac{3}{4}} + \Omega_{125}f_{n+1} + \Omega_{126}f_{n+\frac{5}{4}} + \Omega_{127}f_{n+\frac{3}{2}} + \Omega_{128}f_{n+\frac{7}{4}} + \Omega_{129}f_{n+2}\right)\\
y'_{n+\frac{3}{4}} &= y'_n + \frac{3}{4}hy''_n + \frac{\left(\frac{3}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{131}f_n + \Omega_{132}f_{n+\frac{1}{4}} + \Omega_{133}f_{n+\frac{1}{2}} + \Omega_{134}f_{n+\frac{3}{4}} + \Omega_{135}f_{n+1} + \Omega_{136}f_{n+\frac{5}{4}} + \Omega_{137}f_{n+\frac{3}{2}} + \Omega_{138}f_{n+\frac{7}{4}} + \Omega_{139}f_{n+2}\right)\\
y'_{n+1} &= y'_n + hy''_n + \frac{(h)^2}{2!}y'''_n + h^3\left(\Omega_{144}f_n + \Omega_{142}f_{n+\frac{1}{4}} + \Omega_{143}f_{n+\frac{1}{2}} + \Omega_{144}f_{n+\frac{3}{4}} + \Omega_{145}f_{n+1} + \Omega_{146}f_{n+\frac{5}{4}} + \Omega_{147}f_{n+\frac{3}{2}} + \Omega_{148}f_{n+\frac{7}{4}} + \Omega_{149}f_{n+2}\right)\\
y'_{n+\frac{5}{4}} &= y'_n + \frac{5}{4}hy''_n + \frac{\left(\frac{5}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{151}f_n + \Omega_{152}f_{n+\frac{1}{4}} + \Omega_{153}f_{n+\frac{1}{2}} + \Omega_{154}f_{n+\frac{3}{4}} + \Omega_{155}f_{n+1} + \Omega_{156}f_{n+\frac{5}{4}} + \Omega_{157}f_{n+\frac{3}{2}} + \Omega_{158}f_{n+\frac{7}{4}} + \Omega_{159}f_{n+2}\right)\\
y'_{n+\frac{3}{2}} &= y'_n + \frac{3}{2}hy''_n + \frac{\left(\frac{3}{2}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{161}f_n + \Omega_{162}f_{n+\frac{1}{4}} + \Omega_{163}f_{n+\frac{1}{2}} + \Omega_{164}f_{n+\frac{3}{4}} + \Omega_{165}f_{n+1} + \Omega_{166}f_{n+\frac{5}{4}} + \Omega_{167}f_{n+\frac{3}{2}} + \Omega_{168}f_{n+\frac{7}{4}} + \Omega_{169}f_{n+2}\right)\\
y'_{n+\frac{7}{4}} &= y'_n + \frac{7}{4}hy''_n + \frac{\left(\frac{7}{4}h\right)^2}{2!}y'''_n + h^3\left(\Omega_{171}f_n + \Omega_{172}f_{n+\frac{1}{4}} + \Omega_{173}f_{n+\frac{1}{2}} + \Omega_{174}f_{n+\frac{3}{4}} + \Omega_{175}f_{n+1} + \Omega_{176}f_{n+\frac{5}{4}} + \Omega_{177}f_{n+\frac{3}{2}} + \Omega_{178}f_{n+\frac{7}{4}} + \Omega_{179}f_{n+2}\right)\\
y'_{n+2} &= y'_n + 2hy''_n + \frac{(2h)^2}{2!}y'''_n + h^3\left(\Omega_{181}f_n + \Omega_{182}f_{n+\frac{1}{4}} + \Omega_{183}f_{n+\frac{1}{2}} + \Omega_{184}f_{n+\frac{3}{4}} + \Omega_{185}f_{n+1} + \Omega_{186}f_{n+\frac{5}{4}} + \Omega_{187}f_{n+\frac{3}{2}} + \Omega_{188}f_{n+\frac{7}{4}} + \Omega_{189}f_{n+2}\right)
\end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned}
y''_{n+\frac{1}{4}} &= y''_n + \frac{1}{4}hy'''_n + h^2\left(\Omega_{211}f_n + \Omega_{212}f_{n+\frac{1}{4}} + \Omega_{213}f_{n+\frac{1}{2}} + \Omega_{214}f_{n+\frac{3}{4}} + \Omega_{215}f_{n+1} + \Omega_{216}f_{n+\frac{5}{4}} + \Omega_{217}f_{n+\frac{3}{2}} + \Omega_{218}f_{n+\frac{7}{4}} + \Omega_{219}f_{n+2}\right)\\
y''_{n+\frac{1}{2}} &= y''_n + \frac{1}{2}hy'''_n + h^2\left(\Omega_{221}f_n + \Omega_{222}f_{n+\frac{1}{4}} + \Omega_{223}f_{n+\frac{1}{2}} + \Omega_{224}f_{n+\frac{3}{4}} + \Omega_{225}f_{n+1} + \Omega_{226}f_{n+\frac{5}{4}} + \Omega_{227}f_{n+\frac{3}{2}} + \Omega_{228}f_{n+\frac{7}{4}} + \Omega_{229}f_{n+2}\right)\\
y''_{n+\frac{3}{4}} &= y''_n + \frac{3}{4}hy'''_n + h^2\left(\Omega_{231}f_n + \Omega_{232}f_{n+\frac{1}{4}} + \Omega_{233}f_{n+\frac{1}{2}} + \Omega_{234}f_{n+\frac{3}{4}} + \Omega_{235}f_{n+1} + \Omega_{236}f_{n+\frac{5}{4}} + \Omega_{237}f_{n+\frac{3}{2}} + \Omega_{238}f_{n+\frac{7}{4}} + \Omega_{239}f_{n+2}\right)\\
y''_{n+1} &= y''_n + hy'''_n + h^2\left(\Omega_{241}f_n + \Omega_{242}f_{n+\frac{1}{4}} + \Omega_{243}f_{n+\frac{1}{2}} + \Omega_{244}f_{n+\frac{3}{4}} + \Omega_{245}f_{n+1} + \Omega_{246}f_{n+\frac{5}{4}} + \Omega_{247}f_{n+\frac{3}{2}} + \Omega_{248}f_{n+\frac{7}{4}} + \Omega_{249}f_{n+2}\right)\\
y''_{n+\frac{5}{4}} &= y''_n + \frac{5}{4}hy'''_n + h^2\left(\Omega_{251}f_n + \Omega_{252}f_{n+\frac{1}{4}} + \Omega_{253}f_{n+\frac{1}{2}} + \Omega_{254}f_{n+\frac{3}{4}} + \Omega_{255}f_{n+1} + \Omega_{256}f_{n+\frac{5}{4}} + \Omega_{257}f_{n+\frac{3}{2}} + \Omega_{258}f_{n+\frac{7}{4}} + \Omega_{259}f_{n+2}\right)\\
y''_{n+\frac{3}{2}} &= y''_n + \frac{3}{2}hy'''_n + h^2\left(\Omega_{261}f_n + \Omega_{262}f_{n+\frac{1}{4}} + \Omega_{263}f_{n+\frac{1}{2}} + \Omega_{264}f_{n+\frac{3}{4}} + \Omega_{265}f_{n+1} + \Omega_{266}f_{n+\frac{5}{4}} + \Omega_{267}f_{n+\frac{3}{2}} + \Omega_{268}f_{n+\frac{7}{4}} + \Omega_{269}f_{n+2}\right)\\
y''_{n+\frac{7}{4}} &= y''_n + \frac{7}{4}hy'''_n + h^2\left(\Omega_{271}f_n + \Omega_{272}f_{n+\frac{1}{4}} + \Omega_{273}f_{n+\frac{1}{2}} + \Omega_{274}f_{n+\frac{3}{4}} + \Omega_{275}f_{n+1} + \Omega_{276}f_{n+\frac{5}{4}} + \Omega_{277}f_{n+\frac{3}{2}} + \Omega_{278}f_{n+\frac{7}{4}} + \Omega_{279}f_{n+2}\right)\\
y''_{n+2} &= y''_n + 2hy'''_n + h^2\left(\Omega_{281}f_n + \Omega_{282}f_{n+\frac{1}{4}} + \Omega_{283}f_{n+\frac{1}{2}} + \Omega_{284}f_{n+\frac{3}{4}} + \Omega_{285}f_{n+1} + \Omega_{286}f_{n+\frac{5}{4}} + \Omega_{287}f_{n+\frac{3}{2}} + \Omega_{288}f_{n+\frac{7}{4}} + \Omega_{289}f_{n+2}\right)
\end{aligned}\right\} \quad (8)$$

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$$\left.\begin{aligned}
y'''_{n+\frac{1}{4}} &= y'''_n + h\left(\Omega_{311} f_n + \Omega_{312} f_{n+\frac{1}{4}} + \Omega_{313} f_{n+\frac{1}{2}} + \Omega_{314} f_{n+\frac{3}{4}} + \Omega_{315} f_{n+1} + \Omega_{316} f_{n+\frac{5}{4}} + \Omega_{317} f_{n+\frac{3}{2}} + \Omega_{318} f_{n+\frac{7}{4}} + \Omega_{319} f_{n+2}\right)\\
y'''_{n+\frac{1}{2}} &= y'''_n + h\left(\Omega_{321} f_n + \Omega_{322} f_{n+\frac{1}{4}} + \Omega_{323} f_{n+\frac{1}{2}} + \Omega_{324} f_{n+\frac{3}{4}} + \Omega_{325} f_{n+1} + \Omega_{326} f_{n+\frac{5}{4}} + \Omega_{327} f_{n+\frac{3}{2}} + \Omega_{328} f_{n+\frac{7}{4}} + \Omega_{329} f_{n+2}\right)\\
y'''_{n+\frac{3}{4}} &= y'''_n + h\left(\Omega_{331} f_n + \Omega_{332} f_{n+\frac{1}{4}} + \Omega_{333} f_{n+\frac{1}{2}} + \Omega_{334} f_{n+\frac{3}{4}} + \Omega_{335} f_{n+1} + \Omega_{336} f_{n+\frac{5}{4}} + \Omega_{337} f_{n+\frac{3}{2}} + \Omega_{338} f_{n+\frac{7}{4}} + \Omega_{339} f_{n+2}\right)\\
y'''_{n+1} &= y'''_n + h\left(\Omega_{341} f_n + \Omega_{342} f_{n+\frac{1}{4}} + \Omega_{343} f_{n+\frac{1}{2}} + \Omega_{344} f_{n+\frac{3}{4}} + \Omega_{345} f_{n+1} + \Omega_{346} f_{n+\frac{5}{4}} + \Omega_{347} f_{n+\frac{3}{2}} + \Omega_{348} f_{n+\frac{7}{4}} + \Omega_{349} f_{n+2}\right)\\
y'''_{n+\frac{5}{4}} &= y'''_n + h\left(\Omega_{351} f_n + \Omega_{352} f_{n+\frac{1}{4}} + \Omega_{353} f_{n+\frac{1}{2}} + \Omega_{354} f_{n+\frac{3}{4}} + \Omega_{355} f_{n+1} + \Omega_{356} f_{n+\frac{5}{4}} + \Omega_{357} f_{n+\frac{3}{2}} + \Omega_{358} f_{n+\frac{7}{4}} + \Omega_{359} f_{n+2}\right)\\
y'''_{n+\frac{3}{2}} &= y'''_n + h\left(\Omega_{361} f_n + \Omega_{362} f_{n+\frac{1}{4}} + \Omega_{363} f_{n+\frac{1}{2}} + \Omega_{364} f_{n+\frac{3}{4}} + \Omega_{365} f_{n+1} + \Omega_{366} f_{n+\frac{5}{4}} + \Omega_{367} f_{n+\frac{3}{2}} + \Omega_{368} f_{n+\frac{7}{4}} + \Omega_{369} f_{n+2}\right)\\
y'''_{n+\frac{7}{4}} &= y'''_n + h\left(\Omega_{371} f_n + \Omega_{372} f_{n+\frac{1}{4}} + \Omega_{373} f_{n+\frac{1}{2}} + \Omega_{374} f_{n+\frac{3}{4}} + \Omega_{375} f_{n+1} + \Omega_{376} f_{n+\frac{5}{4}} + \Omega_{377} f_{n+\frac{3}{2}} + \Omega_{378} f_{n+\frac{7}{4}} + \Omega_{379} f_{n+2}\right)\\
y'''_{n+1} &= y'''_n + h\left(\Omega_{381} f_n + \Omega_{382} f_{n+\frac{1}{4}} + \Omega_{318} f_{n+\frac{1}{2}} + \Omega_{384} f_{n+\frac{3}{4}} + \Omega_{385} f_{n+1} + \Omega_{386} f_{n+\frac{5}{4}} + \Omega_{387} f_{n+\frac{3}{2}} + \Omega_{388} f_{n+\frac{7}{4}} + \Omega_{389} f_{n+2}\right)
\end{aligned}\right\} \quad (9)$$

Therefore, to determine the unknown coefficients of ω, we consider $\omega_{\tau i} = \mathrm{X}^{-1}\mathrm{M}$.

Likewise, the unknown coefficients of Ω is given by $\Omega_{\tau i} = \mathrm{X}^{-1}E$

The Numerical Properties of the Numerical Scheme

The basic properties of the new scheme were numerically analyzed according to Skwame et al. (2024). These properties are order and error constant, consistency, zero-stable, convergence and linear stability of the method.

Order and Error Constant

We consider the linear operator $L[y(x_n);h]$, along with Corollaries 2 and 3, to determine the order and error constant of the new method.

Preposition 2

The linear operator $L[y(x_n);h]$, associated with the local truncation error of the new method, is denoted as $C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$.

Proof

The linear difference operators associated with the new method are given by

$$\left.\begin{aligned}
L[y(x_n);h] &= y\left(x_n + \frac{1}{4}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y\left(x_n + \frac{1}{2}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y\left(x_n + \frac{3}{4}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y(x_n + h) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y\left(x_n + \frac{5}{4}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y\left(x_n + \frac{3}{2}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y\left(x_n + \frac{7}{4}h\right) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)\\
L[y(x_n);h] &= y(x_n + 2h) - \left(\alpha_{\frac{1}{4}}\left(x_n + \frac{1}{4}h\right) + \alpha_1(x_n + h) + \alpha_{\frac{3}{2}}\left(x_n + \frac{3}{2}h\right) + \alpha_2(x_n + 2h) + h^4\sum_{i=0}^{2}(\beta_i(x)f_{n+i} + \beta_2(x)f_{n+2})\right)
\end{aligned}\right\} \quad (10)$$

The local truncation error of the new method is derived by assuming $y(t)$ to be sufficiently differentiable. Expanding $y(x_n + qh)$ and $y(x_n + ih)$ about x_n using a Taylor series yields:

$L_{\frac{1}{4}}[y(x_n);h] = (-2..2175\times10^{-07})$, $L_{\frac{1}{2}}[y(x_n);h] = (-3.2795\times10^{-06})$, $L_{\frac{3}{4}}[y(x_n);h] = (-1.3228\times10^{-05})$, $L_1[y(x_n);h] = (-3.3986\times10^{-05})$, $L_{\frac{5}{4}}[y(x_n);h] = (-6.9487\times10^{-05})$, $L_{\frac{3}{2}}[y(x_n);h] = (-1.2373\times10^{-04})$, $L_{\frac{7}{4}}[y(x_n);h] = (-1.9840\times10^{-04})$, $L_2[y(x_n);h] = (-2.8618\times10^{-04})$

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Proof

Expanding Equation (10) with the aid of Corollary 2 and grouping like terms in powers of h, gives

$L_{\frac{1}{4}}[y(x_n);h] = (-2..2175 \times 10^{-07})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{\frac{1}{2}}[y(x_n);h] = (-3.2795 \times 10^{-06})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{\frac{3}{4}}[y(x_n);h] = (-1.3228 \times 10^{-05})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{1}[y(x_n);h] = (-3.3986 \times 10^{-05})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{\frac{5}{4}}[y(x_n);h] = (-6.9487 \times 10^{-05})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{\frac{3}{2}}[y(x_n);h] = (-1.2373 \times 10^{-04})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{\frac{7}{4}}[y(x_n);h] = (-1.9840 \times 10^{-04})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

$L_{2}[y(x_n);h] = (-2.8618 \times 10^{-04})C_{07} h^{07} y^{07}(t_n) + 0(h^{11})$

Consistency

A numerical scheme is said to be consistent if it has an order of convergence greater than or equal to zero, i.e., $(p \geq 1)$. Thus, our new schemes are consistent, since the orders are 5.

Zero Stability

A numerical scheme is said to be Zero-stable for any well behaved initial value problem provided if

i. all roots of $\rho(r)$ lies in the unit disk, $|r| \leq 1$

ii. any roots on the unit circle $(|r| \leq 1)$ are simple

Setting Equation (10) equal to zero and solving for z yields z = 1, confirming that the method is zero-stable.

Convergence

The necessary and sufficient conditions for a numerical scheme to be convergent are consistency and zero stability. Since the new scheme satisfies both consistency and zero stability, it is therefore convergent.

Linear Stability

The set of complex values λh for which all solutions of the test problem $y'''' = -\lambda^4 y$ will remain limited as $n \to \infty$ (Aloko et al., 2024) marks the region of absolute stability of a numerical system.

Applying the test equation $y^{(k)} = \lambda^{(k)} y$ helps to derive the idea of A-stability following (Raymond et al., 2023).

to yield

$$Y_m = \mu(z) Y_{m-1}, \; z = \lambda h \tag{11}$$

where $\mu(z)$ is the amplification matrix given by

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4 \eta^{(0)}\right)^{-1} \left(\xi^1 - z\eta^{(1)} - z^4 \eta^{(1)}\right) \tag{12}$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for of the method is given by

$$\zeta = -\frac{\begin{pmatrix} 131589567585z^8 - 3468986319486z^7 + 42210644799840z^6 - 412492607896852z^5 + 2664153236504256z^4 \\ -13438340522021184z^3 - 44356479052392192z^2 - 95879531652710400z + 94389581905920000 \end{pmatrix}}{\begin{pmatrix} 80015040000z^8 - 1739755584000z^7 + 22504039488000z^6 - 205094550528000z^5 + 1368577244160000z^4 \\ - 6636767477760000z^3 + 22368364462080000z^2 - 47194790952960000z + 47194790952960000 \end{pmatrix}} \tag{13}$$

Numerical Examples

This section evaluates and validates the effectiveness of the derived methods by applying them to solve selected higher-order initial value problems represented by equation (1). To enable a meaningful comparison with results from the literature, the step size h is varied rather than kept constant. The following notations were used in the subsequent tables.

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ES means Exact Solution

CS means Computed Solution

AENS means Abstalute Error in New Scheme

AES20 means Skwame et al., (2020);

AES21 means Sabo et al. (2021);

AEAO18 means Adeyeye and Omar (2018);

AEAO20 means Aizenofe and Olaoluwa (2020)

AEA16 means Akinfenwa et al. (2016);

AEFO17 means Familua and Omole (2017).

Example 1 Consider the Second Order Oscillatory Real-Life Problem (Dynamic Problem)

A 10 kilogram mass is attached to a spring having a spring constant of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(x)=5\sin x$. Find the subsequent motion of the mass $\left(x:0.10\le x\le 1.00\right)$ if the force due to air resistance is $90\left(\frac{dy}{dx}\right)N$.

Applying the procedure, where $m=10,\ k=140,\ a=90\ and\ F(t)=5\sin x$

example 1 reduces to

$$dsolver\left(\left\{\frac{d^2y}{dx^2}+9\frac{dy}{dx}+14y(x)=\frac{1}{2}\sin(x),\ y(0)=0,\ y'(0)=-1\right\}\right) \tag{18}$$

with the exact solution of (4.1) is given by,

$$y(x)=\frac{1}{500}\left(-90e^{-2x}+99ex+13\sin x-9\cos x\right) \tag{19}$$

Source (Skwame et al., 2020; Sabo et al., 2021).

Example 2

Let's investigate a differential equation describing oscillatory behavior up to the third order.

$$y'''(x)=3\sin(x)=0,\quad y(0)=1,\ y'(0)=0,\ y''(0)=-2 \tag{20}$$

with the exact solution given by

$$y(x)=3\cos(t)+\frac{x^2}{2}-2 \tag{21}$$

Source: (Adeyeye & Omar, 2018; Aizenofe & Olaoluwa, 2020)

Example 3

Consider the highly stiff system of fourth order oscillatory problem

$$y^{iv}(x)=4y''(x),\ y(0)=1,\ y'(0)=3,\ y''(0)=0,\ y'''(0)=16 \tag{22}$$

with exact solution given by

$$y(x)=1-v+2\exp(2x)-2\exp(-2x) \tag{23}$$

Source: (Akinfenwa et al., 2016; Familua & Omole, 2017).

Results

Table 1 Showing the numerical results (Skwame et al., 2020; Sabo et al., 2021)

x	**ES**	**CS**	**AENS**	**AES21**	**AES20**
0.1	-0.06436205154552458248	-0.06436205153777350685	7.7511e-12	2.0453e-10	4.4268e-09
0.2	-0.08430720522644774945	-0.08430720521677549486	9.6723e-12	4.8485e-10	2.2383e-08
0.3	-0.08405225313390041905	-0.08405225312996529403	3.9351e-12	6.6174e-10	3.5865e-08
0.4	-0.07529304213333374810	-0.07529304213221919545	1.1146e-12	7.2649e-10	4.2157e-08
0.5	-0.06357063960355798563	-0.06357063960508161767	1.5236e-12	7.1295e-10	4.2895e-08
0.6	-0.05142117069384508163	-0.05142117069633407740	2.4890e-12	6.5550e-10	4.0288e-08
0.7	-0.03993052956438697070	-0.03993052956738402335	2.9971e-12	5.7884e-10	3.6051e-08
0.8	-0.02949865862803573900	-0.02949865863097370062	2.9380e-12	4.9808e-10	3.1287e-08
0.9	-0.02021269131259124546	-0.02021269131532159058	2.7304e-12	4.2140e-10	2.6618e-08
1.0	-0.01202699425403169607	-0.01202699425643041008	2.3987e-12	3.5257e-10	2.2352e-08

See: (Skwame et al., 2020; Sabo et al., 2021).

Table 2 Showing the numerical results for example 2

x	ES	CS	ENS	AEAO18	AEAO20
0.100	0.99001249583407729830	0.99001249583407729828	8.0000e-18	1.7282e-12	4.8906e-10
0.200	0.96019973352372489340	0.96019973352372489333	3.1899e-16	6.3179e-12	3.2663e-09
0.300	0.91100946737681805890	0.91100946737681805886	2.0048e-15	1.4295e-11	1.0296e-08
0.400	0.84318298200865524840	0.84318298200865524823	7.0916e-15	2.5020e-11	2.3509e-08
0.500	0.75774768567111814840	0.75774768567111814812	1.8540e-14	3.8928e-11	4.4764e-08
0.600	0.65600684472903489170	0.65600684472903489137	4.0216e-14	5.5360e-11	7.5847e-08
0.700	0.53952656185346527880	0.53952656185346527833	7.6851e-14	7.4644e-11	1.1844e-07
0.800	0.41012012804149626280	0.41012012804149626223	1.3400e-13	9.6128e-11	1.7411e-07
0.900	0.26982990481199336940	0.26982990481199336883	2.1796e-13	1.2002e-10	2.4429e-07
1.000	0.12090691760441915220	0.12090691760441915146	3.3577e-13	1.4570e-10	3.3028e-07

See: (Adeyeye & Omar, 2018; Aigbiremhon & Omole, 2020).

Table 3 Showing the numerical results for example 3 when h=0.003125

x	**ES**	**CS**	**ENS**
0.003125	1.00937508138036727920	1.00937508138036727920	0.0000(00)
0.006250	1.01875065104675294860	1.01875065104675294860	0.0000(00)
0.009375	1.02812719730424913310	1.02812719730424913310	0.0000(00)
0.001250	1.03750520849609617210	1.03750520849609617210	0.0000(00)
0.015625	1.04688517302275858900	1.04688517302275858900	0.0000(00)
0.018750	1.05626757936100329750	1.05626757936100329750	0.0000(00)
0.021875	1.06565291608298078600	1.06565291608298078600	0.0000(00)
0.025000	1.07504167187531003060	1.07504167187531003060	0.0000(00)
0.028125	1.08443433555816787740	1.08443433555816787740	0.0000(00)
0.031250	1.09383139610438364350	1.09383139610438364350	0.0000(00)

See: (Akinfenwa et al., 2016; Familua & Omole, 2017).

Discussion

The table 1 shows the numerical result for the second-order oscillatory problem, highlighting the errors from the proposed new scheme (AENS) and those from previous studies (Skwame et al., 2020; Sabo et al., 2021). The results demonstrate that new scheme achieves significantly smaller absolute errors compared to the methods of (Skwame et al., 2020; Sabo et al., 2021). This indicates superior accuracy of the new scheme in approximating the motion of the mass. The errors in the new scheme remain consistently low across all computations, showcasing its reliability and precision for solving dynamic problems of this nature.

The table 2 for example 2 also shows the numerical result for a third-order oscillatory differential equation. It highlights the errors from the new scheme and those from previous methods, including (Adeyeye & Omar, 2018; Aizenofe, & Olaoluwa, 2020). The results reveal that the new scheme consistently achieves exceptionally low absolute errors, with AENS values significantly smaller than those of the other methods. This demonstrates the superior accuracy and stability of the new scheme in solving high-order oscillatory problems, emphasizing its effectiveness in providing reliable approximations. The trend of increasing errors in prior methods as computations progress further underlines the precision of the new approach.

Finally, Tables 3 and 4, for example 3 display the numerical results for a highly stiff fourth-order oscillatory problem at two different step sizes, $h = 0.003125$ and $h = 0.01$. In both cases, the Exact Solution and Computed Solution are identical, indicating that the new scheme produces highly accurate solutions. The Absolute Error in New Scheme (AENS) is consistently zero across all time points for both step sizes, suggesting that the computed solution matches the exact solution without error. Furthermore, the table 5 show the absolute errors for previous methods, Akinfenwa et al. (2016) and Familua and Omole (2017), where the errors are significantly larger than those in the new scheme, emphasizing the precision and effectiveness of the proposed approach in solving stiff oscillatory systems.

Conclusion

The research focuses on a new numerical method for solving higher-order oscillatory ordinary differential equations (ODEs). A two-step linear multistep method based on a linear block approach is developed. This method aims to improve the accuracy and efficiency of solving such ODEs by considering factors like consistency, convergence, and zero stability. The proposed method is evaluated using a variety of test problems to compare its performance with other existing methods. The results show that the new method offers higher accuracy and computational efficiency, making it suitable for solving complex higher-order oscillatory ODEs. The study concludes that the newly developed two-step linear multistep method based on a linear block approach is an effective tool for solving higher order oscillatory ODEs. The method exhibits good consistency, convergence, and zero stability properties, along with superior computational efficiency. Its accuracy in solving test problems surpasses that of traditional methods, indicating its potential for broader application in solving complex differential equations in various scientific and engineering fields. Further research could focus on extending this method to more complicated systems and real-world applications.

References

Abdulrahim, R. (2021). Four step hybrid block method for the direct solution of fourth order ordinary differential equations. *Int. J. Anal. Appl*., 12(1), 215-229.

Adewale A. J., & Sabo J. (2024). Simulating the dynamics of oscillating differential equations of mass in motion. *International Journal of Development Mathematics*. 1(1), 54-69. https://doi.org/10.62054/ijdm/0101.07.

Adeyefa, E. O., & Kuboye, J.O. (2020). Derivation of new numerical model capable of solving second and third order ordinary differential equations directly. *IAENG International Journal of Applied Mathematics*, 50 (2020), 2-9.

Adeyeye, O., & Omar, Z. (2018). New self-starting approach for solving special third order initial value problems. *Int. J. Pure Appl. Math*., 118(3), 511-517.

Adeyeye, O., & Omar, Z. (2019). Solving fourth order linear initial and boundary value problems using an implicit block method. *Proceedings of the Third International Conference on Computing, Mathematics and Statistics*, 167-177.

Aizenofe, A. A., & Olaoluwa, O. E. (2020). A four-step collocation procedure by means of perturbation term with application to third-order ordinary differential equation. *International Journal of Computer Applications*, *175*(24), 25-36.

Akinfenwa, O. A., Ogunseye, H. A., & Okunuga, S. A. (2016). Block hybrid method for solution of fourth order ordinary differential equations, *Nigerian Journal of Mathematics and Applications*, 25, 148.

Aloko, D. M., Ayinde, A. M., Usman, D. A., & Sabo, J. (2024). An efficient scheme for direct simulation of third and fourth oscillatory differential equations. *International Journal of Development Mathematics,*

1(1), 72-89.
Ayinde, A. M., Ibrahim, S., Sabo, J**.,** & Silas, D. (2023). The physical application of motion using single step block method. *Journal of Material Science Research and Review,* 6(4), 708-719.
Blanka, B. (2019). Oscillation of second-order nonlinear non-canonical differential equations with deviating argument. *Applied Mathematics Letters,* 91, 68-75.
Donald. J. Z., Skwame, Y., Sabo, J., Kwanamu, J. A., & Daniel, S. (2024). On the Numerical Approximation of Higher Order Differential Equation. *Asian Journal of Research and Reviews in Physics*, *8*(1), 1-26. https://doi.org/10.9734/ajr2p/2024/v8i1154.
Duromola M. K. (2022). Single-step block method of p-stable for solving third-order differential equations (IVPs): Ninth Order of Accuracy. *American Journal of Applied Mathematics and Statistics.* 10(1), 4-13.
Elnady, A. O., Newir, A., & Ibrahim, M. A. (2024). Novel approach for solving higher-order differential equations with applications to the Van der Pol and Van der Pol-Duffing equations. *Beni-Suef Univ J. Basic Appl Sci.*, 13-29. https://doi.org/10.1186/s43088-024-00484-y
Familua, A. D. & Omole, E. O. (2017). Five points mono hybrid point linear multistep method for solving nth order ordinary differential equations using power series function, *Asian Research Journal of Mathematics*, 3**,** 14, 15. DOI: 10.9734/ARJOM/2017/3119.
Folaranmi, R. O., Ayoade, A. A. & Latunde, T. (2021). A Fifth-order hybrid block integrator for third-order initial value problems. *Çankaya University, Journal of Science and Engineering,* 18(2), 87-100.
Kashkari, B. S. H. & Alqarni, S. (2019). Optimization of two-step block method with three hybrid points for solving third order initial value problems. *Journal of Nonlinear Science and Application.* 12: 450-469.
Modebei, M. I. Adeniyi, R. B. Jator, S. N. & Ramos, H. C. (2019). A block hybrid integrator for numerically solving fourth-order initial value problems. *Applied Mathematics and Computation*. 346(1): 680-694.
Sabo, J., Ayinde, A. M., Oyedepo, T., & Ishaq, A. A. (2024). Simulation of Two-Step Block Approach for Solving Oscillatory Differential Equations. *Cumhuriyet Science Journal*, 45(2), 366-378. https://doi.org/10.17776/csj.1345303.
Sabo, J., Kyagya, T. Y., & Vashawa, W. J. (2021). Numerical Simulation of One Step Block Method for Treatment of Second Order Forced Motions in Mass-Spring Systems. *Asian Journal of Research and Reviews in Physics,* 5(2), 1-11.
Sabo, J., Kyagya, T. Y., Ayinde, A. M. & Otaide, I. J. (2022) Mathematical simulation of the linear block algorithm for modeling third-order initial value problems. *BRICS Journal of Educational Research.* 12 (3), 88-96.
Skwame, Y., Donald, J. Z., Kyagya, T. Y., Sabo, J., & Bambur, A. A. (2020). The numerical applications of implicit second derivative on second order initial value problems of ordinary differential equations. *Dutse Journal of Pure and Applied Sciences,* 6(4), 1-14.
Skwame, Y., Zirra, D. J., & John, S. (2024). The numerical application of dynamic problems involving mass in motion governed by higher order oscillatory differential equations. *Physical Science International Journal.* 28(5), 8-31. https://doi.org/10.9734/psij/2024/v28i5845
Ukpebor, L. A., Omole, E. O., & Adoghe, L. O. (2020). An order four numerical scheme for fourth-order initial value problems using Lucas polynomial with application in ship dynamics. *International Journal of Mathematical Research*, 1-28. https://doi.org/10.18488/journal.24.2020.91.28.41
Workineh, Y., Mekonnen, H., & Belew, B. (2024). Numerical methods for solving second-order initial value problems of ordinary differential equations with Euler and Runge-Kutta fourth-order methods. *Front. Appl. Math. Stat.*, 10, 23-49. https://doi.org/10.3389/fams.2024.1360628