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Nilpotent Multigroup and its Properties

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Abstract

The theory of multiset generalizes classical set theory which occur as a result of violating basic property of classical set theory that element has only one frequency or can only belong to a set only once. An algebraic structure that generalized crisp group theory over a multiset was established in (Nazmul et al., 2013) and many of its properties have been explored. In this paper, central series of multigroup is a finite chain of normal subgroups computed via commutator submultigroups which aid the study of nilpotent groups and their properties in the multigroup framework. Therefore, it was established that a multigroup is nilpotent if and only if it has a central series generated either via commutator or centre of multigroups. And for every commutative nilpotent multigroup, the nilpotency is one (1). The nilpotency class of the root set is equivalent to the nilpotency of the nilpotent multigroups.

Keywords: Multigroup, Commutator subgroups, center of multigroup, Nilpotent, Central series

Introduction

The notion of group that has central series was study in classical group by Zeng 2011; Baer 1940; Anthony et al., 2017 and Michael & Ibrahim 2019 show that p – group is a nilpotent group of nilpotency class 2, a finite group is nilpotent if and only if its center is not equal to the identity element and for any non – trivial nilpotent group G, the order of its commutator subgroup is equal to the nilpotency class of G. Since multiset is a generalization of classical set then multigroup came on board which is a non-classical algebraic structure over a multiset framework that generalized group theory and has been explored in diverse area since 2013 by Nazmul et al., 2013 which present properties of multigroup and commutative/abelian multigroups; Ejegwa & Ibrahim, 2016; Awolola 2019 consider cyclic multigroup family; Adamu & Ibrahim 2020 study strongly invariant submultigroups; Awolola & Ibrahim 2016; Awolola and Ibrahim 2016 established homomorphism of multigroup and first, second and third isomorphism was presented and Michael et al., 2024 examined and redefined center of multigroups that has multiset as its root set. The concept of group was deployed in this paper to study multigroup that is captured on series which has not been studied. Central series is a chain of normal subgroups generated via commutator of submultigroup. Nilpotent multigroup is a group that has central series which is studied via commutator submultigroups and center of a multigroup as defined in (Michael et al., 2024) which capture the notion of set with multiplicity greater than or equal to one.

Preliminary

This section presents fundamental definition on concept of multigroup that will be used in the subsequent sections of this paper.

Definition 2.1 (Singh et al., 2007) Let X be a set. Multiset A drawn over X is a cardinal valued function D_G defined as $D_A : X \longrightarrow \mathbb{N} = \{x: x \geq 0\}$. For each $x \in X, D_A(x)$ represent the number of frequency of element x in the multiset A drawn from $X = \{x_1, x_2, x_3, \dots, x_n\}$ which can be written as $A = [x_1, x_2, x_3, \dots, x_n]_{m_1, m_2, m_3, \dots, m_n}$ such that x_i appears m_i times, $i = 1, 2, \dots., n$ in the multisets A.

Definition 2.2 (Singh et al., 2007) Let A and B be two multisets over X, A is called a submultiset of B written as $A \subseteq B$ if $D_G(x) \leq D_G(x), \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and is denoted as $A \subset B$.

Definition 2.3 (Singh et al., 2007) A multiset A is regular multiset over X if all its objects occur with the same multiplicity and the common multiplicity is called its height otherwise A is irregular

For example, $A = [a^4, b^4, c^4]$ is a regular multiset of height 4

Definition 2.4 (Syropoulos, 2001) Let A and B be two multisets over X. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively are defined by the rules that for any objects $x \in X$,

i. $D_{A\cap B}(y^{-1}) = D_A(y) \wedge D_B(y)$

ii. $D_{A\cup B}(y^{-1}) = D_A(y) \vee D_B(y)$

where $\wedge$ and $\vee$ are minimum (infimum) and maximum (supremum) respectively

Definition 2.5 (Nazmul et al., 2013) Let X be a group. A multiset G over X is called a multigroup over X if the count function C_G satisfies the following conditions;

$$D_G(xy^{-1}) \geq D_G(x) \wedge D_G(y), \qquad \forall x, y \in X,$$

The set of all multigroups over a group X is denoted by $MG(X)$, since

$$D_G(e) = D_G(yy^{-1}) \geq D_G(y) \wedge D_G(y^{-1})$$

Definition 2.6 (Nazmul et al., 2013) Suppose X be a group and $G \in MG(X)$, then multugroup G is a commutative multigroup if it satisfies $D_G(xy) = D_G(yx), x, y \in X$.

Definition 2.7 (Nazmul et al., 2013) Let X be a group and $G \in MG(X)$, then set $X \coloneqq \{x \in X : D_G(x) \geq 0\}$ is known as the root set or support of G. Hence, $C_G(x) = 0$ if x is not an object in G and conversely if $D_G(x) > 0$.

Definition 2.8 (Nazmul et al., 2013) Let X be a group and $G \in MG(X)$, then $G^{-1} \in MG(X)$ is defined as $D_{G^{-1}}(x) = D_G(x^{-1}), \forall\, x \in X$.

Definition 2.9 (Ejegwa & Ibrahim, 2020) Let $G \in MG(X)$. A submultiset H of G is called a submultigroup if H form a multigroup and is denoted as $H \sqsubseteq G$. A submultigroup H of G is called a proper submultigroup denoted by $H \sqsubset G$, if $H \sqsubseteq G$ and $G \neq H$.

Definition 2.10 (Ejegwa & Ibrahim, 2017) A submultigroup, H of a multigroup $G \in MG(X)$ is a normal submultigroup if and only if $h \in H_* : D_H(xhx^{-1}) \geq D_H(h), \forall x \in G_*$.

Definition 2.11 (Michael et al., 2024) Let G be a multigroup over a group X then the center of G, denoted by $C(G)$ is the submultiset of G generated by $D(G) \coloneqq D_G(x)$, if $D_G(xy) = D_G(yx), \forall y \in X$.

Definition 2.12 (Michael and Ibrahim 2023) Let $G \in MG(X)$ and $A, B \sqsubseteq G$. Then the commutator of A and B is the submultigroup denoted by $[A, B]$ and

$$D_{[A,B]}(x) \coloneqq \bigvee \{D_A(a) \wedge D_B(b)\} : x = [a, b]\}.$$

Theorem 2.13 (Ejegwa & Ibrahim, 2017) Let $A, B \in MG(X)$ such that $A \subseteq B$, then A is a normal submultigroup of B if and only if

i. $D_A([x, y]) = D_A(xyx^{-1}y^{-1}) \geq D_A(x), \forall x, y \in X$

ii. $D_A([x, y]) = D_A(xyx^{-1}y^{-1}) = D_A(e), \forall x, y \in X$

Where the identity element of X is represented by e.

Definition 2.14 (Ejegwa & Ibrahim, 2017) Let $A \in MG(X)$. Then

i. $A_* = \{x \in X : D_A(x) > 0\}$

ii. $A^* = \{x \in X : D_A(x) = D_A(e)\}$

where e is the identity element of X, A_* and A^* are subgroups of X.

Definition 2.15 (Ejegwa and Ibrahim, 2017) Let X be a group. For any submultigroup $A \sqsubseteq G \in MG(X)$, the submultiset yA of $G, \forall y \in X$ is $D_{yA}(x) \coloneqq D_A(y^{-1}x)$, for all $x \in A_*$ is called the left commultiset of A. Also, submultiset Ay of $G, \forall y \in X$ $D_{Ay}(x) \coloneqq D_A(xy^{-1})$, for all x an element of the root set

Definition 2.16 (Ejegwa & Ibrahim, 2017) Let B be a submultigroup of multigroup G over group X then the left commultiset is equivalent to the right commultiset i.e., $yB = By, \forall x \in X$.

Results

Definition 3.1 Let $\boldsymbol{G \in MG(X)}$, $\boldsymbol{Z_0(G) \coloneqq G}$, $\boldsymbol{Z_1(G) \coloneqq [G, G]}$ and in general $\boldsymbol{Z_n(G) \coloneqq [Z_{n-1}(G), G]}$, for $\boldsymbol{n \geq 1}$. Then chain $\boldsymbol{G = Z_0(G) \supseteq Z_1(G) \supseteq \cdots \supseteq Z_n(G) \supseteq \cdots}$ such that $\boldsymbol{Z_i(G) \trianglelefteq G}$, and $\boldsymbol{\frac{Z_i(G)}{Z_{i+1}(G)} \subseteq C(\frac{G}{Z_{i+1}(G)})}$ for $\boldsymbol{i \geq 0}$ is known as the descending central chain of $\boldsymbol{G}$.

Example

Let $X = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta\}$ be a group under matrix multiplication for

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \beta_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\beta_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta_6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

$$A = [\beta_1{}^{10}, \beta_2{}^5, \beta_3{}^7, \beta_4{}^5, \beta_5{}^5, \beta_6{}^5, \beta_7{}^7, \beta_8{}^8] \in MG(X).$$ Then
$D_{\gamma[G]}(\beta_1) = D_G(\beta_1),\ D_{\gamma[G]}(\beta_3) = D_G(\beta_5)$
Hence $Z_1(G) = D_{\gamma[G]}(x) = [\beta_1{}^{10}, \beta_3{}^5]. Z_2(G) = D_{\gamma[G]}(x) = [\beta_1{}^{10}]$.
Therefore, $G = Z_0(G) \supseteq Z_1(G) \supseteq Z_2(G)$.

Theorem 3.1 Given that $G \in MG(X)$. Then $Z_n(G) \subseteq Z_{n-1}(G), \forall\, n \geq 1$.

Proof Let $G \in MG(X)$ such that G has a central chain $G = Z_0(G) \supseteq Z_1(G) \supseteq \cdots \supseteq Z_2(G) \supseteq \cdots$. By induction on $n \geq 0$. Let $x \in X$. If $x \notin \{[a,b], a, b \in X\}$, then $D_{[G,G]}(x) = 0 \leq D_G(x)$. Otherwise, $D_{[G,G]}(x) > 0$, then we have

$$\begin{aligned} D_{[G,G]}(x) &= \vee \{\{D_G(x) \wedge D_G(b)\} : x = [a,b]\} \\ &= \vee \{\{D_G(a^{-1}) \wedge D_G(b^{-1}) \wedge D_G(a) \wedge D_G(b)\} : x = [a,b]\} \\ &\leq \vee \{D_G(a^{-1}b^{-1}ab)\} \\ &= D_G(x). \end{aligned}$$

Thus, $D_{[G,G]}(x) \leq D_G(x), x \in X$. Since $[G,G] \sqsubseteq G$,

$$D_{Z_1(G)}(x) = D_{[G,G]}(x) \leq D_G(x) = D_{Z_0(G)}(x), x \in X\}.$$

Therefore, for $n = 1$ we have $D_{Z_1(G)}(x) \leq D_{Z_0(G)}(x), x \in X$.
Now, suppose $Z_k(G) \subseteq Z_{k-1}(G)$ for some $k \geq 1$, then

$$D_{Z_{k+1}(G)}(x) \leq D_{Z_k(G)}(x), \forall x \in \{[a,b], a, b \in X\}.$$

Hence, $Z_{k+1}(G)$equivalent to $[Z_k(G), G]$ and $[Z_k(G), G] \subseteq [Z_{k-1}(G), G] = Z_k(G)$

Theorem 3.2 Let $G \in MG(X)$, then every submultigroups G on the central series of G is a normal submultigroups.
Proof Let $G \in MG(X)$, and the descending chain of submultigroups of G, for $x \in X$ we have $D_G(x) = D_{Z_0(G)}(x) \supseteq D_{Z_1(G)}(x) \supseteq \cdots \supseteq D_{Z_n(G)}(x) \supseteq \cdots$
Let, $H \sqsubseteq G$ and $D_{Z_i(G)}(x) = D_H(x), x \in X$, then $\forall h \in H_*, D_H(xh) = D_H(hx), x \in X$
$\Longrightarrow\ C_H(xhx^{-1}) = C_H(h)$. Clearly, H is a normal submultigroups in multigroup G.

Let $G_1, \dots, G_n$ be submultigroups of $G \in MG(X)$, we define the submultigroups $[G_1, \dots, G_n] \in MG(X)$ inductively as $[G_1, \dots, G_n] = [[G_1, \dots, G_{n-1}], G_n], n \geq 3$.

Theorem 3.3 Let $G \in MG(X)$ such that $G_1, \dots, G_{n+1} \sqsubseteq Z_K(G) :\ C_{G_i}(x) = C_G(x), \forall x \in X$ for $k + 1\ (k \geq 0)$ different values of i, then $[G_1, \dots, G_{n+1}] \sqsubseteq Z_K(G)$.
Proof Let $G \in MG(X)$ and for $i > 0$, $G_i \sqsubseteq G$, for $n > 0$ then we show that for $k + 1 (k \geq 0)$, $D_{[G_1, \dots, G_{n+1}]}(x) \leq D_{Z_K(G)}(x), \forall x \in \{[a,b], a, b \in X\}$.
By induction hypothesis on n, for $n = 0$, $D_{G_1}(x) = D_{Z_0(G)}(x) = D_G(x), x \in \{[a,b], a, b \in X\}$.
Let G_i be a submultigroups of G, then if $x \in \{[a,b], a, b \in X\}$ we have

$$D_{[G_i,G]}(x) = D_{[G,G_i]}(x) \leq D_G(x) = D_{Z_0(G)}(x),$$

and $D_{[G,G]}(x) = D_{Z_1(G)}(x)$, which shows that the result holds for $n = 1$ and for all possible values of k. Now, let $G_1, \dots, G_{n+1}$ be submultigroups of G such that $G_i = G$ for $k + 1 (k \geq 0)$ distinct values of i. Firstly, let $k = 0$, if $G_{n+1} = G$, then since $[G_1, \dots, G_n]$ is a normal in G, we have

$$D_{[G_1, \dots, G_{n+1}]}(x) = D_{[[G_1, \dots, G_n], G]}(x), x \in \{[a,b], a, b \in X\} \leq D_G(x) = D_{Z_0(G)}(x).$$
$$\Longrightarrow\ [G_1, \dots, G_{n+1}] = [[G_1, \dots, G_n], G] \sqsubseteq G = Z_0(G)$$

If $D_{G_{n+1}}(x) \neq D_G(x)$ then, by induction hypothesis, $D_{[G_1, \dots, G_n]}(x) \leq D_{Z_0(G)}(x), x = [a,b]$ and therefore, $D_{[G_1, \dots, G_{n+1}]}(x) \leq D_{[Z_0(G), G_{n+1}]}(x) \leq D_{Z_0(G)}(x), x \in \{[a,b], a, b \in X\}$. Next, let $k \geq 1$ and $D_{G_{n+1}}(x) \neq D_G(x)$, then by induction hypothesis $D_{[G_1, \dots, G_n]}(x) \leq D_{Z_k(G)}(x), x = [a,b]$
Also, we have $D_{[G_1, \dots, G_{n+1}]}(x) \leq D_{Z_k(G)}(x)$. Suppose $D_{G_{n+1}}(x) = D_G(x)$ then, again

$$D_{[G_1, \dots, G_n]}(x) \leq D_{Z_{k-1}(G)}(x).$$

Thus, $D_{[G_1, \dots, G_{n+1}]}(x) \leq D_{[Z_{k-1}(G), G]}(x) = D_{Z_k(G)}(x), x \in \{[a,b], a, b \in X\}$.
Definition 3.5 Let $G \in MG(X)$, then the central series of G is a chain of normal submultigroups

$$D_G(x) = D_{G_0}(x) \geq D_{G_1}(x) \geq \cdots \geq D_{G_n}(x) = D_G(e), \text{for } n \geq 0,$$
$$D_{G_{n+1}}([a,b]) \geq D_{G_n}(a) \wedge D_G(b).$$

Definition 3.6 Nilpotent Multigroup

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Let $G \in MG(X)$, then G is nilpotent if it has central series

$$D_{(G)}(x) = D_{Z_0(G)}(x) \geq D_{Z_1(G)}(x) \geq \cdots \geq D_{Z_c(G)}(x) = D_G(e).$$

Then G is nilpotent of class c.

Remark: Let G be a nilpotent multigroup. The nilpotency class of G is equivalent nilpotent class of the root set.

Examples 3.7

i. Let $X = \{1, -1, i, -i\}$ be a group, $X' = \{1\}$ and $G = [1^4, -1^3, \pm i^2] \in MG(X)$. Then $1 = \{[1,1], [1,-1], [1,i], [1,-i], [i,i], [-i,-i]\}$

$$D_{\gamma_{[G]}}(1) = \vee \{D_G(1), D_G(-1), D_G(i), D_G(-i), D_G(i), D_G(-i)\} = D_G(1)$$

$D_{[\gamma_{[G]},G]}(1) = D_G(1)$, Hence $D_{[\gamma_{[G]},G]}(x) = D_{\gamma_{[G]}}(1)$

Therefore G is a nilpotent multigroup.

ii. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13),$$
$$\rho_5 = (12),\ X' = \{\rho_0, \rho_1, \rho_2\}$$

Then $G = [{\rho_0}^7, {\rho_1}^4, {\rho_2}^4, {\rho_3}^3, {\rho_4}^3, {\rho_5}^3] \in MG(X)$and $D_{\gamma_{[G]}}(x) = [{\rho_0}^5, {\rho_1}^3, {\rho_2}^3]$. then $D_{[G,\gamma_{[G]}]}(x) = [{\rho_0}^5, {\rho_1}^3, {\rho_2}^3] \neq D_G(e)$.

Hence G is not a nilpotent multigroup.

Theorem 3.4 Let $G \in MG(X)$, then G is nilpotent if and only if G has a central series.

Proof Suppose $G \in MG(X)$ and $D_G(x) = D_{G_0}(x) \geq D_{G_1}(x) \geq \cdots \geq D_{G_n}(x) = D_G(e),\ \forall x \in X$ be the descending central series for multigroup G. Then by induction on i $D_{Z_i(G)}(x) \leq D_{G_i}(x)$, for $x \in \{[a,b], a, b \in X\}, i = 0, 1, \dots, n$. We have

$D_{Z_0(G)}(x) = D_G(x) = D_{G_0}(x)$, for $i = 0$ and $D_{Z_1(G)}(x) = D_{[G,G_0]}(x) \leq D_{G_1}(x)$.

If $D_{Z_i(G)}(x) \leq D_{G_i}(x)$, for some $i = 0, 1, \dots, n-1$ then we have

$$D_{Z_{i+1}(G)}(x) = D_{[Z_i(G),G]}(x), x \in \{[a,b], a, b \in X\} \leq D_{[G_i,G]}(x) \leq D_{G_{i+1}}(x).$$

Thus $D_{Z_i(G)}(x) \leq D_{G_i}(x)$ for $i = 0, 1, \dots, n$. Therefore we get

$$D_G(e) \leq D_{Z_n(G)}(x) \leq D_{G_n}(x) = D_G(e), x \in \{[a,b], a, b \in X\}.$$

Hence, G is nilpotent of class at most n. Conversely, if G is nilpotent of class c, then $D_G(x) = D_{Z_0(G)}(x) \geq D_{Z_{1(G)}}(x) \geq \cdots \geq D_{Z_c(G)}(x) = D_G(e), \forall x \in \{[a,b], a, b \in X\}$

is the central series of G.

Theorem 3.5 Let X be a nilpotent group and $G \in MG(X)$. Then G has a central series

Proof Let $G \in MG(X)$ and X has a central series whose length is c, then for $0 \leq i \leq c$, we define $C_{G_i}(x) = C_G(e), x \in Z_i(G)$ and 0 if the case is not so. Then

$$D_G(x) = D_{G_0}(x) \geq D_{G_1}(x) \geq \cdots \geq D_{G_c}(x)$$

is a finite chain of submultigroups of G over X such that $(G_i)_* \leq (Z_i(G))_*$ for each i. Obviously, $D_{G_c}(x) = D_G(x), x \in (G_i)_*$, where c is the nilpotency class of X. Now, let $a, b \in X$ and $0 \leq i \leq c$. If $a \notin (Z_i(G))_*$, then we have $\{D_{G_i}(a) \wedge D_G(b)\} = 0 \leq D_{G_{i+1}}([a,b])$.

Conversely, let $a \in (Z_i(G))_*$ then $D_G([a,b]) > 0$ and therefore

$$\{D_{G_i}(a) \wedge D_G(b)\} = \{D_G(a) \wedge D_G(b)\} \leq D_G([a,b]) = D_{G_{i+1}}([a,b]).$$

Thus, the central chain $D_G(x) = D_{G_0}(x) \geq D_{G_1}(x) \geq \cdots \geq D_{G_c}(x) = D_G(e)$ of G and therefore G is nilpotent of class c.

Theorem 3.6 Every submultigroup of a nilpotent multigroup is normal if and only if X is commutative.

Proof Let $N \sqsubseteq G \in MG(X)$, then G has a central series. Let, $H \sqsubseteq G$ such that

$$D_N(xy) \geq D_N(x) \wedge D_N(y^{-1}), x, y \in N_*$$
$$D_N(xn\,x^{-1}) = D_N(n)$$
$$\forall n \in N_*, D_N(xn) = D_N(nx), \qquad x \in X$$

which implies that $xy = yx, \forall x, y \in X$. Therefore, N is a normal submultigroup in G.

Theorem 3.7 Let G be a nilpotent multigroup over a group X. If $H \sqsubseteq G$ then H is nilpotent.

Proof Let H be a submultigroup of a nilpotent multigroup G. So $D_{Z_n(H)}(x) \leq D_{Z_n(G)}(x),\ \forall n \geq 0$. Therefore, if G is nilpotent of class c, then H is nilpotent of class at most c.

Theorem 3.8 Let G be a nilpotent multigroup and $A, B \sqsubseteq G \in MG(X)$ then $A \circ B$ is nilpotent.

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Proof If G be a nilpotent multigroup and $A, B \sqsubseteq G$ then A has a finite chain such that $D_A(x) = D_{A_0}(x) \geq D_{A_1}(x) \geq \cdots \geq D_{A_n}(x) = D_A(e), x \in \{[a,b], a, b \in X\}$
Similarly, for multigroup B, $D_{(A\circ B)}(t) = \vee \{D_A(y) \wedge D_B(z), : t = yz\}$ for each $n \geq 0$, $D_{Z_n(A\circ B)}(x) \leq D_{[G,G]}(e)$, if $x \in \{[a,b], a, b \in X\}$

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