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Introducing Symmetric Rhotrices: Properties and Applications

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Abstract
This study introduces symmetric rhotrices as an extension of symmetric matrices, establishing their foundational structure and defining key operations. We explore their properties, particularly in function analysis, through Hessian rhotrices, which aid in classifying critical points—except in cases where the determinant is zero.
The research examines fundamental concepts such as positive definiteness, diagonalization, eigenvectors, and eigenvalues, using the Hessian Rhotrix of R_4, an even-dimensional rhotrix, as a case study. The spectral theorem is employed to determine diagonalizability, analyze vector transformations, and demonstrate vector alignments within R_4. Additionally, we apply the second derivative test to classify critical points.
This work contributes to the development of rhotrix algebra and its potential applications in fields such as machine learning, quantum mechanics, and optimization theory

Keywords: Symmetric Rhotrix, Eigenvalues, Diagonalization, Hessian Rhotrix, Positive Definiteness, Spectral Theory.

Introduction

The study of matrix algebra has played a crucial role in numerical analysis, data science, and physics. Theoretical advancements in matrix representations have led to the development of new algebraic structures, such as rhotrices, which extend traditional matrices. Ajibade (2003) introduced rhotrices as a mathematical structure positioned between 2×2 and 3×3 matrices, extending earlier work on matrix-tertions and matrix-noitrets by Atanassov (1998). Ajibade's rhotrix is defined in its heart-oriented form as:

$$\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \tag{1}$$

where $h(R)$ is referred to as the "heart" of the rhotrix. Later, Isere (2018) introduced even-dimensional rhotrices, which differ from Ajibade's definition by omitting the heart term, making them "heartless rhotrices":

$$\left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle \tag{2}$$

This structural difference has led to the development of alternative multiplication methods for rhotrices. Notably, Sani (2004) proposed a row-column multiplication method, while Utoyo (2023) introduced a co-minor's approach for even-dimensional rhotrices.
Given that matrices are typically classified as square or rectangular, while rhotrices inherently maintain equal numbers of rows and columns, we introduce the concept of symmetric rhotrices: Rhotrices that remain unchanged when transposed. These structures exhibit three fundamental properties:

i. Real eigenvalues,
ii. Orthogonal eigenvectors, and
iii. Diagonalizability (also guaranteed by spectral theory).

To the best of our knowledge, symmetric rhotrices have not been extensively studied in prior literature. This work aims to establish their algebraic properties, demonstrate their role in function analysis, and explore real-world applications.

1. **Preliminaries:**

The following are germane to the study:

Definition 2.1 (Symmetric Rhotrix)

A symmetric rhotrix is a rhotrix that is equal to its transpose. That is, given a rhotrix R, $R = R^T$ where transposition involves flipping the entries over the major vertical axis while preserving the diagonal. For example, given

$$R = \left\langle \begin{matrix} & a & \\ b & & d \\ & e & \end{matrix} \right\rangle, \text{ then}, R^t = \left\langle \begin{matrix} & e & \\ d & & b \\ & a & \end{matrix} \right\rangle \tag{3}$$

A rhotrix is symmetric if $R = R^T$, meaning if and only if b is equal to d.

Definition 2.2 (Similarity Transformation of a Rhotrix)

A similarity transformation of a rhotrix S by an invertible rhotrix R is define as $T = R^{-1}SR$ where R^{-1} is the inverse of R and S represents the transformed rhotrix, S is the original rhotrix and R is an invertible rhotrix.

$$R = \left\langle \begin{matrix} & a & \\ b & & d \\ & e & \end{matrix} \right\rangle \; and \; S = \left\langle \begin{matrix} & f & \\ g & & i \\ & j & \end{matrix} \right\rangle,$$

The rhotrix R has an inverse R^{-1}, provided that the determinant $\det(R) = ps - qr \neq 0$.

$$R^{-1} = \frac{1}{ae - bd} \left\langle \begin{matrix} & e & \\ -b & & -d \\ & a & \end{matrix} \right\rangle$$

we calculate T using the similarity transformation formula

$$T = R^{-1}SR.$$

to obtain

$$R^{-1}S = \frac{1}{ae - bd} \left\langle \begin{matrix} & e & \\ -b & & -d \\ & a & \end{matrix} \right\rangle \left\langle \begin{matrix} & f & \\ g & & i \\ & j & \end{matrix} \right\rangle$$

therefore

$$\mathrm{T} = \frac{1}{ae - bd} \left\langle \begin{matrix} & a(ef - bh) + b(af - dh) & \\ a(eg - bj) + b(ag - dj) & & d(ef - bh) + e(af - dh) \\ & d(eg - bj) + e(ag - dj) & \end{matrix} \right\rangle \tag{4}$$

If R is an orthogonal rhotrix $\left(R^T = R^{-1}\right)$, then the similarity transformation preserves symmetry, that is, T remains symmetric.

Definition 2.3 (A Self-Adjoint Rhotrix)

A self-adjoint rhotrix (or Hermitian rhotrix) on a finite-dimensional real or complex inner product space is a type of linear operator

$$T : R \to R$$

that satisfies:
$\langle T(x), y\rangle - \langle x, T(y)\rangle$ for all vectors x, y in the space R, where, $\langle \, , \rangle$ denotes the inner product. In rhotrix representation, a linear operator T acting on $\Re^n$ or (C^n) is self-adjoint if and only if its rhotrix representation S satisfies:

$$S^T = S \text{ for real rhotrices,}$$
$$S^\tau = S \text{ for complex rhotrices,}$$

where S^τ is the conjugate transpose. This means that S must be symmetric (or Hermitian in the complex case). Given

$$R = \left\langle \begin{matrix} & a & \\ b & & d \\ & e & \end{matrix} \right\rangle \, and \; S = \left\langle \begin{matrix} & f & \\ g & & i \\ & j & \end{matrix} \right\rangle,$$

For S to be self-adjoint, we need $S^T = S$ which implies that $h = g$. Thus, we assume $R^{-1}S$ is

$$R^{-1}S = \frac{1}{ae - bd} \left\langle \begin{matrix} & e & \\ -b & & -d \\ & a & \end{matrix} \right\rangle \left\langle \begin{matrix} & f & \\ g & & i \\ & j & \end{matrix} \right\rangle$$

Obtaining $T = R^{-1}SR$
The transpose is given as:

$$\mathrm{T} = \frac{1}{ae - bd} \left\langle \begin{matrix} & a(ef - bh) + b(af - dh) & \\ a(eg - bj) + b(ag - dj) & & d(ef - bh) + e(af - dh) \\ & d(eg - bj) + e(ag - dj) & \end{matrix} \right\rangle$$

$$= \frac{1}{ae - bd} \left\langle \begin{matrix} & aef - abg - baf + bdg & \\ aeg - abj - bag + baj & & def - dbg - eaf + edg \\ & \deg - dbj - eag + eaj & \end{matrix} \right\rangle \quad (5)$$

For T to be self-adjoint, it must satisfy $T^T = T$. This does not always hold unless R is orthogonal.
We solve this system for x_i and y_i in terms of each other, which gives the eigenvector corresponding to λ_i. Explicitly, we solve for the eigenvectors.

$$v_i = \left\langle \begin{matrix} x_i \\ y_i \end{matrix} \right\rangle$$

Results

A symmetric rhotrix of R_2 is a 2×2 rhotrix. R satisfies $R^T = R,$ meaning, it is equal to its transpose. This plays a fundamental role in numerical analysis, physics and machine learning.

Definition 3.1 (Fundamental Properties of Symmetric Rhortices)}
The eigenvalues of a rhotrix transformation can be determined by solving the characteristic equation, (Zhang, 2021):

$$\det(R - \lambda I) = 0$$

where I is the identity rhotrix and λ is an eigenvalue.
For a given rhotrix R_4:

$$R=\left\langle\begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & & g & h \\ & i & j & k & \\ & & l & & \end{array}\right\rangle$$

the characteristic equation is derived as follows:

$$\det(R-\lambda \mathrm{I})=\left\langle\begin{array}{ccccc} & & \lambda_1 & & \\ & \lambda_2 & \lambda_3 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \lambda_6 & \lambda_5 & \\ & & \lambda_4 & & \end{array}\right\rangle=0$$

Solving for λ gives the eigenvalues of R_4 as

$$\left\langle\begin{array}{ccccc} & & \frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & \frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & \frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & \frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & \frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array}\right\rangle \quad (6)$$

Since the rhotrix is symmetric, the discriminant $\Delta=(a-l)^2+4eh,\ \Delta=(b-k)^2+4di,\ \Delta=(c-j)^2+4fg,$ ensures that all eigenvalues are real, making symmetric rhotrices always diagonalizable.

Definition 3.2 (Eigenvectors Corresponding to the Eigenvalues Are Orthogonal.)
(Aminu, 2010) . An eigenvector is a non-zero vector that, when a linear transformation (represented by a rhotrix) is applied to it, changes only in scale and not in direction. For each eigenvalue λ_i, the corresponding eigenvector v_i is found by solving

$$(R-\lambda_i I)v_i=0$$

If

$$R=\left\langle\begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & & g & h \\ & i & j & k & \\ & & l & & \end{array}\right\rangle$$

$$\lambda_1=\frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2};\ \lambda_2=\frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2};\ \lambda_3=\frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2}$$

$$\lambda_4=\frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2};\ \lambda_5=\frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2};\ \lambda_6=\frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2}.$$

Substituting λ_i into $R_4-\lambda_i$, we obtain

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$$\left\langle \begin{array}{ccccc} & & a-\frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & b-\frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & c-\frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & j-\frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & k-\frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & l-\frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array} \right\rangle v_i = 0$$

To proof that eigenvectors corresponding to distinct eigenvalues are orthogonal, we consider the rhotrix R_4 and its eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, we solve

$$(R_4 - \lambda_i)v_{ij} = 0$$

Substituting $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, to find the corresponding eigenvectors $v_1, v_2, v_3, v_4, v_5, v_6$.
We solve

$$\left\langle \begin{array}{ccccc} & & a-\frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & b-\frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & c-\frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & j-\frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & k-\frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & l-\frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array} \right\rangle \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & & x_3 \\ & & \\ y_3 & & y_2 \\ & y_1 & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle$$

This represents a system of linear equations. Using the element-wise multiplication method which is associative and the rhotrix-scalar multiplication property, which is commutative, we can move the scalar freely. The systems of equations then simplifies to:

$$R^T = \left\langle \begin{array}{ccccc} & & -\frac{e}{a-\lambda_1} & & \\ & -\frac{d}{b-\lambda_2} & -\frac{f}{c-\lambda_3} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\frac{j-\lambda_6}{f} & -\frac{k-\lambda_5}{d} & \\ & & -\frac{l-\lambda_4}{e} & & \end{array} \right\rangle \tag{7}$$

After rewriting x_i in terms of y_i.

Definition 3.3 (Symmetric rhotrices are always Diagonalizable)
(Usaini Salisu, 2014), introduced a classic problem in linear algebra, the diagonalization problem in terms of rhotrices (RDP). Since the eigenvectors of a symmetric rhotrix form an orthogonal basis, every symmetric rhotrix is diagonalizable. This implies that we can express R_4 as

$$R_4 = R_H \Delta R_H^{-1}$$

If

$$\Delta = \left\langle \begin{array}{ccccc} & & \lambda_1 & & \\ & \lambda_2 & \lambda_3 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \lambda_6 & \lambda_5 & \\ & & \lambda_4 & & \end{array} \right\rangle$$

Multiplying R_H with Δ and then by R_H^{-1} we recover R_4, confirming that R_4 is diagonalizable. From the definition of similarity transformation, the rhotrix R_4 is similar to a diagonal rhotrix. If a rhotrix is symmetric, then it is diagonalizable using a rhotrix whose columns are its eigenvectors. Hence,

$$\left\langle \begin{array}{ccccc} & & -\dfrac{e}{a-\lambda_1} & & \\ & -\dfrac{d}{b-\lambda_2} & -\dfrac{f}{c-\lambda_3} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\dfrac{j-\lambda_6}{f} & -\dfrac{k-\lambda_5}{d} & \\ & & -\dfrac{l-\lambda_4}{e} & & \end{array} \right\rangle \left\langle \begin{array}{ccccc} & & \lambda_1 & & \\ & \lambda_2 & \lambda_3 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \lambda_6 & \lambda_5 & \\ & & \lambda_4 & & \end{array} \right\rangle \otimes \frac{1}{\dfrac{l-\lambda_4}{a-\lambda_1} - 1 - \dfrac{k-\lambda_5}{b-\lambda_2} - 1 - \dfrac{j-\lambda_6}{c-\lambda_3} - 1} \left\langle \begin{array}{ccccc} & & -\dfrac{l-\lambda_4}{e} & & \\ & -\dfrac{k-\lambda_5}{d} & -\dfrac{j-\lambda_6}{f} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\dfrac{c-\lambda_3}{f} & -\dfrac{b-\lambda_2}{d} & \\ & & -\dfrac{a-\lambda_1}{e} & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & \lambda_1 & & \\ & \lambda_2 & \lambda_3 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \lambda_6 & \lambda_5 & \\ & & \lambda_4 & & \end{array} \right\rangle = R_4 \tag{8}$$

This means that applying the transformation $R_H \Delta R_H^{-1}$ preserves the properties of the symmetric rhotrix while simplifying computations.

Since diagonalization represents a rhotrix in terms of its eigenvalues and eigenvectors, it simplifies various operations, such as rhotrix exponentiation and solving differential equations involving rhortices.

Applications in Symmetric Rhotrix

Symmetric rhotrices are essential in the study of abstract structure, where their real eigenvalues, orthogonality and computational efficiency enable applications in eigendecomposition, spectral clustering, stability analysis, and so on.

Theorem 1. Spectral Theory

Spectral theory states that every symmetric rhotrix can be diagonalized by an orthogonal rhotrix. Thus, the eigenvalues of a symmetric rhotrix are always real and its eigenvectors are linearly independent. The set of all the eigenvalues of a rhotrix is called a spectrum. Also, the eigenvalue-eigenvector pairs tell us in which direction a vector is distorted after a given linear transformation. This is shown in the following figure.

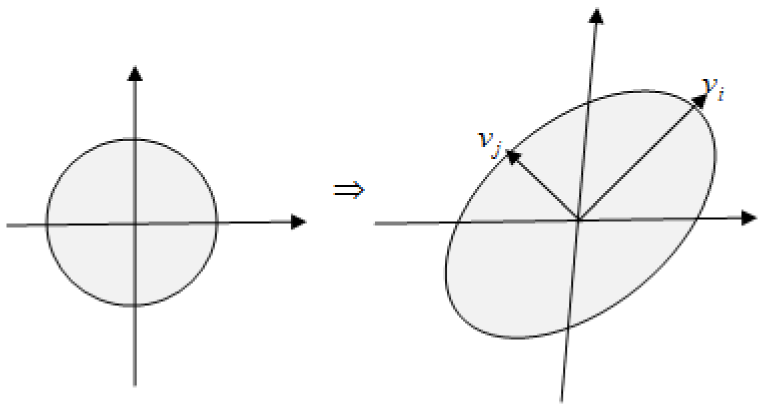

Figure 1. Comparing an un-stretched vector (left) to a stretched vector (right).

After transformation, in the direction of v_i, the figure is stretched a lot. But in the direction of v_j it's not stretched very much. Hence, if

$$Q=\left\langle\begin{array}{ccccc} & & -\frac{e}{a-\lambda_1} & & \\ & -\frac{d}{b-\lambda_2} & -\frac{f}{c-\lambda_3} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\frac{j-\lambda_6}{f} & -\frac{k-\lambda_5}{d} & \\ & & -\frac{l-\lambda_4}{e} & & \end{array}\right\rangle \text{ and } \left\langle\begin{array}{ccccc} & & \frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & \frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & \frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & \frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & \frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array}\right\rangle$$

then, the symmetric rhotrix R_4 is diagonalizable as $R_4 = Q\Lambda Q^T$. That is,

$$
\left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & & g & h \\ & i & j & k & \\ & & l & & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & -\dfrac{e}{a-\lambda_1} & & \\ & -\dfrac{d}{b-\lambda_2} & -\dfrac{f}{c-\lambda_3} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\dfrac{j-\lambda_6}{f} & -\dfrac{k-\lambda_5}{d} & \\ & & -\dfrac{l-\lambda_4}{e} & & \end{array} \right\rangle \otimes
$$

$$
\left\langle \begin{array}{ccccc} & & \dfrac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & \dfrac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & \dfrac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \dfrac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & \dfrac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & \dfrac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array} \right\rangle \otimes
$$

$$
\left\langle \begin{array}{ccccc} & & -\dfrac{e}{a-\lambda_1} & & \\ & -\dfrac{d}{b-\lambda_2} & -\dfrac{f}{c-\lambda_3} & 1 & \\ 1 & 1 & & 1 & 1 \\ & 1 & -\dfrac{j-\lambda_6}{f} & -\dfrac{k-\lambda_5}{d} & \\ & & -\dfrac{l-\lambda_4}{e} & & \end{array} \right\rangle^T \tag{9}
$$

R_4 is diagonalizable because it has real eigenvalues, orthogonal eigenvectors and satisfies $Q\Lambda Q^T$. Therefore, in the process of the diagonalization $R_4 = Q\Lambda Q^T$, Q sends a vector from the standard basis to the eigenvectors, Λ scales it, and then Q^T sends the vector back to the standard basis. From the perspective of the vector, the coordinate system is aligned with the standard basis with the eigenvectors. For example,

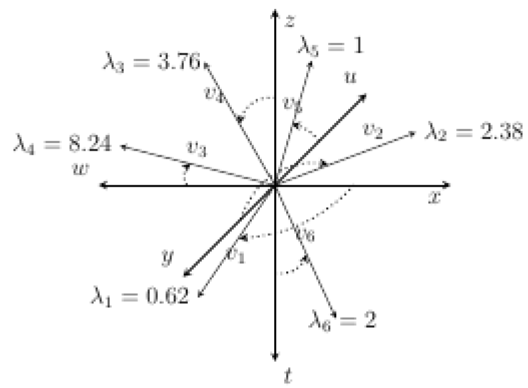

Figure 2. Alignment of the basis of R_4

The rhotrix used in the alignment of the basis is:

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 5 & 1 & 3 & \\ 1 & 0 & & 0 & 1 \\ & 3 & 2 & 7 & \\ & & 2 & & \end{array} \right\rangle = v \left\langle \begin{array}{ccccc} & & 0.62 & & \\ & 8.24 & 1 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & 2 & 3.76 & \\ & & 2.38 & & \end{array} \right\rangle v^{-1}$$

Symmetry and Positive-definiteness

Positive-definiteness refers to a specific property of a rhotrix that indicates how it behaves when applied to vectors. To determine the definiteness of a symmetric (or Hermitian) rhotrix, we evaluate its eigenvalues. The definiteness of a rhotrix can be classified based on the signs of its eigenvalues. They are positive definite, positive semi-definite, negative definite, negative semi-definite and definite. Positive-definite rhotrices ensures that a quadratic form $x^T Rx$ has a unique minimum and positive for all non-zero vectors which is critical in convex optimization and in stability analysis. If,

$$R_4 = \left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & & g & h \\ & i & j & k & \\ & & l & & \end{array} \right\rangle \text{and it's eigenvalues are} \left\langle \begin{array}{ccccc} & & \frac{(a+l)+\sqrt{(a+l)^2-4(al-eh)}}{2} & & \\ & \frac{(b+k)+\sqrt{(b+k)^2-4(bk-di)}}{2} & \frac{(c+j)+\sqrt{(c+j)^2-4(cj-fg)}}{2} & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & \frac{(c+j)-\sqrt{(c+j)^2-4(cj-fg)}}{2} & \frac{(b+k)-\sqrt{(b+k)^2-4(bk-di)}}{2} & \\ & & \frac{(a+l)-\sqrt{(a+l)^2-4(al-eh)}}{2} & & \end{array} \right\rangle$$

then, R_4 is positive-definite since $a+l>0,\ b+k>0,\ c+j>0,$ and $\langle al-eh>0,\ bk-di>0,\ cj+fg>0\rangle$

Hessian Rhotrices and Function Behaviour

Brown (2014) explores the relationship between curvature, concavity and the eigenvalues of the Hessian Matrix providing insights into its role in determining the nature of critical points. This offers a good computation and application of a symmetric rhotrix. The theorem of the second derivative test for functions of several variables offers analyzes on function behavior of rhotrices. The test helps to determine whether a critical point of a function is a local minimum, local maximum or a saddle point. Formally, let $f:\Re^n \to \Re$ be a function, the Hessian is defined as

$$H(x) = \langle h_{ij}(x) \rangle = \left\langle \frac{d^2 f(x)}{dx_i dx_j} \right\rangle.$$

The Hessian matrix is widely used to analyze function concavity, local extrema, and critical points (Omolehin, 2006). The Hessian rhotrix extends this concept by storing second-order partial derivatives in a rhotrix structure. For a function $f(x,y)$, the Hessian rhotrix is defined as:

$$H(x_i, y_j) = \left\langle \begin{array}{ccccc} & & \frac{d^2 f}{dx_1^2} & & \\ & \frac{d^2 f}{dx_2^2} & \frac{d^2 f}{dx_3^2} & \frac{d^2 f}{dy_3 dx_3} & \\ \frac{d^2 f}{dx_1 dy_1} & \frac{d^2 f}{dx_2 dy_2} & & \frac{d^2 f}{dy_2 dx_2} & \frac{d^2 f}{dy_1 dx_1} \\ & \frac{d^2 f}{dx_3 dy_3} & \frac{d^2 f}{dy_3^2} & \frac{d^2 f}{dy_2^2} & \\ & & \frac{d^2 f}{dy_1^2} & & \end{array} \right\rangle$$

Theorem 4.2. The Second Derivative Test

The second derivative test states that:

If all eigenvalues are positive, the function has a local minimum.

If all eigenvalues are negative, the function has a local maximum.
If eigenvalues have mixed signs, the function has a saddle point.
By applying rhotrix algebra, Hessian rhotrices can be used in machine learning (for curvature-aware optimization), economics (for stability analysis), and physics (for energy minimization problems). We use the Hessian rhotrix at this point to determine the nature of the critical point. If the vector of the partial derivatives of a function f is zero at some point x, then f has a critical point at x. The determinant of the Hessian at x is called a discriminant. If the discriminant is zero then x is called a degenerate critical point of f or a called a non-Morse Critical point of f. otherwise it is non-degenerate and called a Morse Critical point. If

$$H(x_i, y_i)\left\langle \begin{array}{ccccc} & & \frac{\partial^2 f}{\partial x_1^2} & & \\ & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3^2} & \frac{\partial^2 f}{\partial x_2 \partial y_2} & \\ \frac{\partial^2 f}{\partial x_1 \partial y_1} & \frac{\partial^2 f}{\partial x_3 \partial y_3} & & \frac{\partial^2 f}{\partial y_3 \partial x_3} & \frac{\partial^2 f}{\partial y_1 \partial x_1} \\ & \frac{\partial^2 f}{\partial y_2 \partial x_2} & \frac{\partial^2 f}{\partial y_2^2} & \frac{\partial^2 f}{\partial y_2^2} & \\ & & \frac{\partial^2 f}{\partial y_1^2} & & \end{array} \right\rangle$$

Consider the functions $f(x,y) = \frac{x^2}{2} - xy + y^2$ or $\frac{5x^2}{2} + 3xy + \frac{7y^2}{2}$ or $\frac{1}{2}x^2 + y^2$. The Hessian is computed as follows:

System i. Given the function $f(x,y) = \frac{x^2}{2} - xy + y^2$, its second-order partial derivatives $\frac{\partial^2 f}{\partial x_1^2} = x_1 + \mathrm{y}_1$.

Differentiate $f(x,y)$, with respect to x twice $\frac{\partial f}{\partial x_1} = x_1 + \mathrm{y}_1$, $\frac{\partial^2 f}{\partial x_1^2} = 1$ $\frac{\partial^2 f}{\partial y_1^2}$. Differentiate $f(x,y)$ with respect to y twice, $\frac{\partial f}{\partial y_1} = x_1 + 2\mathrm{y}_1$, $\frac{\partial^2 f}{\partial y_1^2} = 2$ $\frac{\partial^2 f}{\partial x_1 \partial y_1}$. Differentiate $f(x,y)$ first with respect to x then with respect to y, $\frac{\partial f}{\partial x_1} = x_1 + \mathrm{y}_1$, $\frac{\partial^2 f}{\partial x_1 \partial y_1} = 1$, $\frac{\partial^2 f}{\partial y_1 \partial x_1}$. Differentiate $f(x,y)$ first with respect to y then with respect to x, $\frac{\partial f}{\partial y_1} = x_1 + 2\mathrm{y}_1$, $\frac{\partial^2 f}{\partial y_1 \partial x_1} = 1$. Therefore, the Hessian rhotrix

$$H(x,y) = \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 2 & \end{array} \right\rangle.$$

System ii. Given the function $f(x_2, y_2) = \frac{5x_2^2}{2} + 3x_2 y_2 + \frac{7y_2^2}{2}$, its second-order partial derivatives $\frac{\partial^2 f}{\partial x_2^2} = 5x_2 + 3\mathrm{y}_2$. Differentiate $f(x_2, y_2)$, with respect to x_2 twice $\frac{\partial f}{\partial x_2} = 5x_2 + 3\mathrm{y}_2$, $\frac{\partial^2 f}{\partial x_2^2} = 5$ $\frac{\partial^2 f}{\partial y_2^2}$. Differentiate $f(x_2, y_2)$ with respect to y_2 twice, $\frac{\partial f}{\partial y_2} = 3x_2 + 7\mathrm{y}_2$, $\frac{\partial^2 f}{\partial y_1^2} = 7$ $\frac{\partial^2 f}{\partial x_2 \partial y_2}$.

Differentiate $f\left(x_2, y_2\right)$ first with respect to x_2 then with respect to y_2, $\frac{\partial f}{\partial x_2} = 5x_2 + 3\mathrm{y}_2$, $\frac{\partial^2 f}{\partial x_2 \partial y_2} = 3$, $\frac{\partial^2 f}{\partial y_2 \partial x_2}$. Differentiate $f\left(x_2, y_2\right)$ first with respect to y_2 then with respect to x_2, $\frac{\partial f}{\partial y_2} = 3x_2 + 7\mathrm{y}_2$, $\frac{\partial^2 f}{\partial y_2 \partial x_2} = 3$. Therefore, the Hessian rhotrix $H\left(x_2, y_2\right) = \left\langle \begin{array}{ccc} & 5 & \\ 3 & & 3 \\ & 7 & \end{array} \right\rangle$.

System iii. Given the function $f\left(x_3, y_3\right) = \frac{1}{2}x_3^2 + y_3^2$, its second-order partial derivatives $\frac{\partial^2 f}{\partial x_3^2} = x_3$. Differentiate $f\left(x_3, y_3\right)$, with respect to x_3 twice $\frac{\partial f}{\partial x_3} = x_3$, $\frac{\partial^2 f}{\partial x_3^2} = 1$ $\frac{\partial^2 f}{\partial y_3^2}$. Differentiate $f\left(x_3, y_3\right)$ with respect to y_3 twice, $\frac{\partial f}{\partial y_3} = 2\mathrm{y}_3$, $\frac{\partial^2 f}{\partial y_3^2} = 2$ $\frac{\partial^2 f}{\partial x_3 \partial y_3}$. Differentiate $f\left(x_3, y_3\right)$ first with respect to x_3 then with respect to y_3, $\frac{\partial f}{\partial x_3} = x_3$ $\frac{\partial^2 f}{\partial x_3 \partial y_3} = 0$, $\frac{\partial^2 f}{\partial y_3 \partial x_3}$. Differentiate $f\left(x_3, y_3\right)$ first with respect to y_3 then with respect to x_3, $\frac{\partial^2 f}{\partial y_3 \partial x_3}. = 2\mathrm{y}_3$, $\frac{\partial^2 f}{\partial y_3 \partial x_3} = 0$. Therefore, the Hessian rhotrix $H\left(x_3, y_3\right) = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 2 & \end{array} \right\rangle$.

Notice that it's symmetric, which is always true for a Hessian rhotrix of a function with continuous second-order partial derivatives. The Hessian rhotrix at a critical point (where the gradient is zero) determines the nature of that critical point. The positive definite operation of the hessian rhotrix is positive definite at a critical point, the critical point is a local minimum. A rhotrix is positive definite if all it's eigenvalues are positive or equivalently, if all its leading principal minors are positive. Negative definite occurs if the Hessian is negative definite at a critical point, the critical point is local maximum. A rhotrix is negative definite if all it's eigenvalues are negative. Indefinite, if the Hessian is indefinite (has both positive and negative eigenvalues), the critical point is a saddle point and Semidefinite if the Hessian is semidefinite (has some zero eigenvalues), however, further analysis is needed to determine the nature of the critical point. Thus, the Hessian rhotrix for the three system becomes

$$H\left(x_i, y_i\right) \left\langle \begin{array}{ccccc} & & 1 & & \\ & 5 & 1 & 3 & \\ 1 & 0 & & 0 & 1 \\ & 3 & 2 & 7 & \\ & & 2 & & \end{array} \right\rangle$$

For all three functions, the Hessian rhotrix at any critical point is positive definite. Therefore, any critical points of these functions are local minima. The images of the functions are shown below:

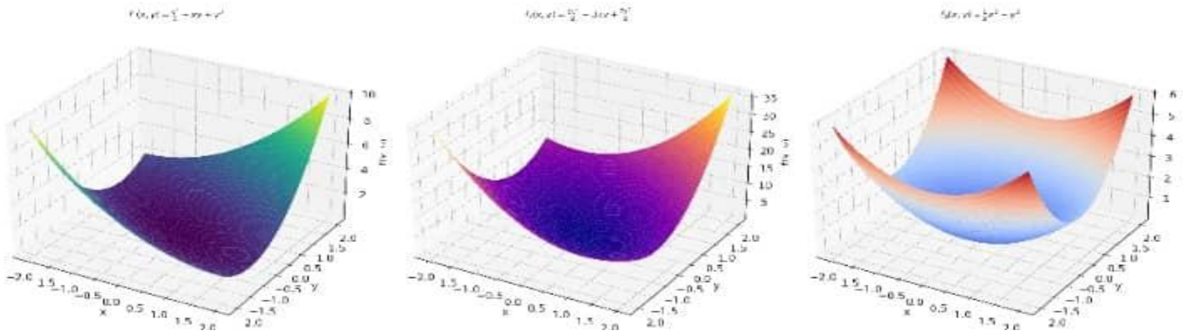

Figure 3. Images of the functions

Rhotrices that are equal to their transpose are called symmetric rhotrices. To transpose a rhotrix, one flips its entries over the major vertical axis, keeping the entries on this axis unchanged. The graphs of the functions when flipped represents a concave surface showing that in all directions, the functions decreases as the eigenvalues remain positive, signifying a flipped growth pattern in the eigenvectors directions as:

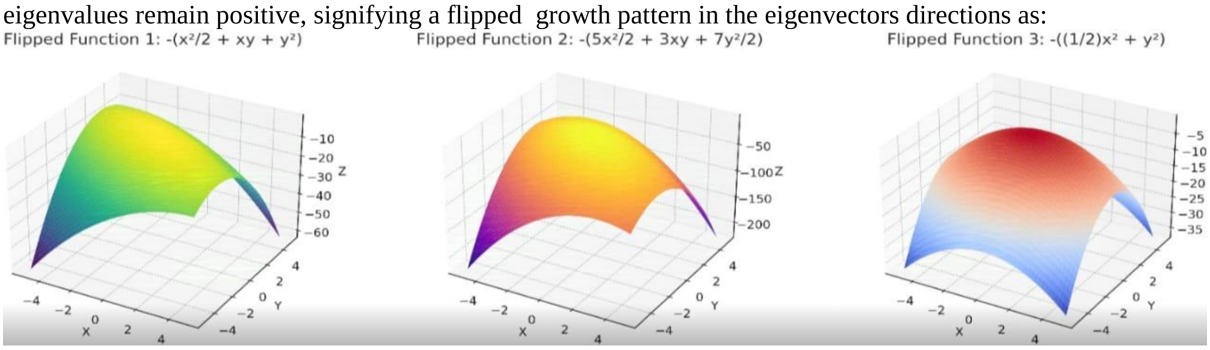

Figure 4. Images of the transposed functions

The relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up through the application of Hessian rhotrices in function analysis. By examining the second-order partial derivatives of a function, the Hessian rhotrix helps classify critical points based on its eigenvalues. When all eigenvalues are negative, the function is concave down at that point, indicating a local maximum. Conversely, when all eigenvalues are positive, the function is concave up, signifying a local minimum. This is crucial in optimization and stability analysis, as the study confirms that positive definite Hessian rhotrices lead to stable minima, while negative definite Hessian rhotrices indicate peak values in function behavior.

Conclusion

This study introduces symmetric rhotrices and explores their algebraic properties, including eigenvalues, eigenvectors, diagonalization, Positive-definitness and Hessian rhotrices. The key findings include: Symmetric rhotrices always have real eigenvalues and eigenvectors that are orthongonal ensuring diagonalizability. Positive-definiteness ensures stability, optimization, and function classification, making it essential in spectral analysis. Hessian rhotrices provide a robust framework for function classification, extending classical second-order tests. Potential applications exist in numerical optimization, machine learning, and quantum mechanics.

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Future work should explore numerical implementations of symmetric rhotrices and their integration into computational frameworks like MATLAB and Python.

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