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A Hybrid Algorithm for Solving Nonlinear Problems

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Abstract
We construct an algorithm for the approximation of the common solution of the set of fixed points of a countable family of closed Bregman quasi-strict pseudo-contractive mappings; set of solutions of a finite system of a generalized mixed equilibrium problems and the set of h-fixed points of a finite family of h-pseudo-contractive mappings. A strong convergence theorem is proved for this algorithm in a reflexive (real) Banach space. An application is provided.

Keywords: Bregman Quasi-Strict Pseudo-Contractive Mappings, System Of Generalized Mixed Equilibrium Problems

Introduction

The study of fixed point theory is indeed an interesting research area for many researchers in the field of analysis and applied mathematics (Rockafeller, 1970; Naraghirad & Yao, 2013; Zalinescu, 2002; Chidume, 2009). The reason for the interest is based on the fact that many problems that occur in physical phenomena are often modeled in the form of fixed point problem as follows $x = Tx$, where T is taken to be any mapping or operator (Reich & Sabach, 2010, Bregman, 1967, Censor, 1981, Kato,1967). Approximation of fixed points of mappings (when it exists) is predominantly done through iterative methods. However, approximation of fixed points of Bregman quasi strictly pseudo-contractive mappings cannot be possible using classical iterative methods, as they will either give a weak convergence or the sequences generating the methods will not be bounded (see Reich & Sabach, 2009, Zegeye & Shahzad, 2014, Ugwunnadi et.al, 2014). We note that some authors have intensively studied this class of Bregman mapping (Wang, 2015; Wang & Wei, 2017; Yongfu & Yongchun, 2015; Ali et.al., 2019). Examples of this class of mapping can be found in (Ugwunnadi et.al., 2014, Zegeye et.al., 2022). In many cases, monotonicity condition of a mapping is very important in fixed point and optimization theories such that it often guarantees convergence. A closely related monotone maps is h −pseudo-contractive maps. This map was originally introduced and studied by Zegeye et.al, 2022 and these mappings has become a motivation for the study of system of generalized mixed equilibrium problems.

Consequently many classical methods are available in the literature (Picard, 1890; Krasnosel'skii, 1955; Mann, 1953, Ishikawa, 1974) for approximating common solutions of several related nonlinear problems. Many of these methods have been successfully applied in approximating fixed point of several mappings and for finding solutions of some nonlinear system of equations respectively. The literature of fixed point theory is heavily domicile in Hilbert space, CAT(0) space and as well as in general Banach spaces, where many valuable problems related to practical problems are generally defined. Two convergence results; weak and strong convergence has always been of great interest to researchers.

Polyak (1964) became the first author to come up with acceleration process of convergence of iteration methods known as inertial-type algorithm. This algorithm is known for solving many smooth convex minimization problems. This discovery has led many authors to combine this method with other classical iterative methods to accelerate the rate of convergence of the sequence generated by the proposed algorithm and some references within (Ali et.al., 2019; Dong et.al., 2018; Ekuma-Okereke & Okoro, 2020; Padcharoen et.al 2020). Zegeye et al. (2022), introduced an algorithm for approximating common solutions of a finite family of generalized mixed equilibrium problems, the set of h-fixed points of a finite family of h-pseudo-contractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings. Their algorithm is as follows: let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} u, x_0 \in K, \\ z_n = P_K \nabla h^*(\nabla h(x_n) - a_n M_n x_n), \\ w_n = P_K \nabla h^*(\nabla h(x_n) - a_n M_n z_n), \\ u_n = G_{H_M}^{h,r_n} \circ G_{H_{M-1}}^{h,r_n} \circ \dots \circ G_{H_2}^{h,r_n} \circ G_{H_1}^{h,r_n} x_n, \\ v_n = R_{S_N}^{h,r_n} \circ R_{S_{N-1}}^{h,r_n} \circ \dots \circ R_{S_2}^{h,r_n} \circ R_{S_1}^{h,r_n} u_n, \\ x_{n+1} = \nabla h^*(a_n \nabla h(u) + b_n \nabla h(x_n) + c_n \nabla h(w_n) + d_n \nabla h(v_n)), \end{cases} \tag{1}$$

where ∇h is the gradient of h on X, $(r_n) \subset [c, \infty)$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are scalar sequence in (0,1) such that $a_n + b_n + c_n + d_n = 1$, then the sequence converges to the common element in real reflexive Banach spaces.

Using the method of inertial algorithm combined with resolvent method, Ali et al, (2019), introduced and studied algorithm for approximating a common fixed point of countable families of quasi Bregman strictly pseudo-contractive mappings and solution of a system of generalized mixed equilibrium problems as follows:

$$\begin{cases} C_1 = X, \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_n = \nabla h^*\big(b_n \nabla h(z_n) + (1 - b_n)\nabla h(T_t z_n)\big), \\ u_n = G^{r_n} y_n, \\ C_{n+1} = \{u \in C_n : d_h(u, u_n) \le d_h(u, y_n) \le d_h(u, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla h(T_t z_n), z_n - u\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^h(x_0), n \in \mathbb{N}. \end{cases} \tag{2}$$

The authors proved that the sequence generated by the algorithm converges to the common element of the set of solutions of a countable family of generalized mixed equilibrium problems, the set of fixed points of a countable families of quasi Bregman strictly pseudo-contractive mappings in real reflexive Banach spaces.

Ugwunnadi et al. (2014), originally introduced a new class of Bregman mapping called quasi-Bregman strictly pseudo-contraction mapping. They constructed a new iterative technique called a hybrid method and proved a strong convergence theorem of a common element in the set of fixed points of a family of a finite family of closed quasi-Bregman strictly pseudo-contraction mappings and common solution to a system of equilibrium problems in reflexive Banach space. Below is their iterative technique:

$$\begin{cases} x_1 = x \in K, \\ y_n = \nabla h^*(\alpha \nabla h(x_n) + (1 - \alpha_n)\nabla h(T_n x_n)), \\ u_{j,n} = Res_{g_j}^h y_n, j = 1, 2, \dots, M, \\ w_n = \nabla h^*(\sum_{j=1}^{M} \beta_{j,n} \nabla h(u_{j,n})), \\ C_{n+1} = \{z \in C_n : d_h(x_n, w_n) \le \frac{1}{1-\lambda}\langle \nabla h(x_n) - \nabla h(T_n x_n), x_n - z\rangle \\ +\langle \nabla h(T_n x_n) - \nabla h(w_n), x_n - z\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^h(x), n \in N, \end{cases} \tag{3}$$

where $T_n = T_n(modN)$ and $\alpha_n \in (0,1)$ satisfying some conditions and $\lambda \in [0,1)$, for each $i = 1, 2, \dots, N$, T_i is uniformly continuous. Then under suitable conditions, the sequence $\{x_n\}$ converges strongly to the solution set Ω.

Chidume et al. (2018), formulated and studied an inertial algorithm for approximating common fixed point of a countable family of relatively nonexpansive mappings. Set $x_0, x_1 \in X$, and define a sequence $\{x_n\}$ by the following algorithm:

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$$\begin{cases} C_0 = X, \\ z_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}((1-\beta)Jz_n + \beta JTz_n), \\ C_{n+1} = u \in C_n: \phi(u, y_n) \le \phi(u, z_n), \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \ge 0, \end{cases} \tag{4}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$, $T: X \to X$ is a relatively nonexpansive map. The authors showed that their sequence converge strongly to a fixed point set of their mapping.

Motivated by the above results, we introduce an inertial algorithm for approximating common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of h-fixed points of a finite family of h-pseudo-contractive mappings and the set of fixed points of countable families of Bregman quasi strictly pseudo-contractive mappings in real reflexive Banach spaces. We prove a strong convergence theorem for it. Consequently we have several applications of our method. Furthermore, an efficient numerical illustration is given to justify the theoretical hypothesis of our results. Our method and approach is considered to generalized many existing problems in the literature.

Methods and Model Formulations

Hypothesis

1. Let K represent a non-empty, closed and convex subset of a reflexive real Banach space X and its dual space X^*;
2. let $h: X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fre'chet differentiable and totally convex on bounded subsets of X;
3. let $T_t: X \to X, \forall t = 1,2, \ldots,$ represent countable family of closed and Bregman quasi-strict pseudo-contractive mappings such that $(I - T_t)$, $t = 1,2, \ldots,$ is demi-closed at the origin;
4. let $S_i: X \to X^*$, $\forall j = 1,2, \ldots N$ represent continuous h-pseudo-contractive mappings;
5. let $\Theta_i: K \times K \to \mathbb{R}, i = 1,2, \ldots M,$ be a continuous bi-functionals satisfying

Condition A;

6. let $A_i: K \to X^*, i = 1,2, \ldots M,$ be a continuous monotone mappings;
7. let $\theta_i: K \to \mathbb{R}, i = 1,2, \ldots M,$ be a real valued functions;
8. let the common solution set be denoted by Ω be a non-empty set, that is
$$\Omega := [\cap_{t=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} GMEP(\Theta_i, A_i, \theta_i))] \cap \left[(\cap_{j=1}^{N} Fix_h(S_j))\right] \neq \emptyset;$$
9. $K \subset int(dom\ h)$, interior domain of the Legendre function.

Step 1: Given the initialization $x_0, x_1 \in K$, compute

$$z_n = x_n + a_n(x_n - x_{n-1}), \tag{5}$$

Step 2: Define the Mann iteration to incorporate Bregman quasi-strict pseudo-contractive mappings given that the generating sequence is the inertial term z_n, and with respect to Bregman distance. Next compute the sequence as follows;

$$y_n = \nabla h^*\big(b_n \nabla h(z_n) + (1 - b_n)\nabla h(T_t z_n)\big), \tag{6}$$

Step 3: Define and compute a resolvent function, composed of finite continuous h-pseudo-contractive defined on the y_n as follows:

$$u_n = G_{H_M}^{h,r_n} \circ G_{H_{M-1}}^{h,r_n} \circ \ldots \circ G_{H_2}^{h,r_n} \circ G_{H_1}^{h,r_n} y_n \tag{7}$$

Step 4: Define and compute a resolvent function, composed of finite system of generalized equilibrium problem involving bi-functionals defined on the u_n as follows;

$$v_n = R_{S_N}^{h,r_n} \circ R_{S_{N-1}}^{h,r_n} \circ \ldots \circ R_{S_2}^{h,r_n} \circ R_{S_1}^{h,r_n} u_n, \tag{8}$$

Step 5: Define the half-closed feasible solution set with respect to the Bregman function as follows:

$$C_{n+1} = \{u \in C_n: d_h(u, v_n) \le d_h(u, u_n) \le d_h(u, y_n) \le d_h(u, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla h(T_t z_n), z_n - u\rangle\} \tag{9}$$

Step 5 Compute the next iterate using the Bregman projection method so that the solution lies in the feasible set as follows:

$$x_{n+1} = P_{C_{n+1}}^{h}(x_0), n \in \mathbb{N}. \tag{10}$$

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If $z_n = y_n = u_n = v_n = x_{n+1} = 0$ for $n = 1$, then stop the process. If not, repeat **Step 1-Step 5** until it converges and the solution found to belong to the common feasible sets Ω.

From the above steps, we now have the coupled algorithm below.

Algorithm 2.1.

Initialize $\{x_0, x_1\}$. Let the sequence $\{x_n\}$ and $\{z_n\}$ be generated by the following algorithm:

$$\begin{cases} C_1 = X, \\ z_n = x_n + a_n(x_n - x_{n-1}), \\ y_n = \nabla h^*\big(b_n \nabla h(z_n) + (1 - b_n)\nabla h(T_t z_n)\big), \\ u_n = G_{H_M}^{h,r_n} \circ G_{H_{M-1}}^{h,r_n} \circ \ldots \circ G_{H_2}^{h,r_n} \circ G_{H_1}^{h,r_n} y_n, \\ v_n = R_{S_N}^{h,r_n} \circ R_{S_{N-1}}^{h,r_n} \circ \ldots \circ R_{S_2}^{h,r_n} \circ R_{S_1}^{h,r_n} u_n, \\ C_{n+1} = \{u \in C_n : d_h(u, v_n) \le d_h(u, u_n) \le d_h(u, y_n) \le d_h(u, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla h(T_t z_n), z_n - u\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^h(x_0), n \in \mathbb{N}, \end{cases} \tag{11}$$

where ∇h is the gradient of h on X, $(r_n) \subset [c, \infty)$, $\{a_n\}$, $\{b_n\} \in (0,1)$ are scalar control sequences.

The following definitions and concept will be used in what follows.

Definition 2.1 *Let h be a convex and Gâteaux differentiable function at x, then the bi-function $d_h(.,.)$: $dom\ h \times int(dom\ h) \to \mathbb{R}^+$ defined by*

$$d_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x\rangle \tag{12}$$

is the Bregman distance induced by the convex function h (see Reich & Sabach, 2010, Bregman, 1967, Censor & Lent, 1981 and the references contain therein).

Definition 2.2 *A map $T: K \to int(dom\ h)$ with respect to a convex function $h: X \to (-\infty, +\infty]$ is called*

1. Bregman quasi-strict pseudo-contraction (shortly,(BQSPC))(see Ugwunnadi et.al 2014) if there exists a constant $\rho \in [0,1)$ and $Fix(T) \neq \emptyset$ such that the following conditions holds
$$d_h(p, Tx) \le d_h(p, x) + \rho d_h(x, Tx), \forall x \in K, \forall p \in Fix(T) \tag{13}$$
2. T is called closed if for any $\{x_n\} \subset K$ with $x_n \to x$ and $Tx_n \to z \in K$ as $n \to \infty$, then $Tx = z$.

Definition 2.3 *A mapping $M: X \to X^*$ is monotone if for all $x, y \in dom(M)$, we have*

$$\langle Mx - My, x - y\rangle \ge 0. \tag{14}$$

Similarly,

Definition 2.4 *$T: X \to X^*$ is said to be h −pseudo-contractive maps if for each $x, y \in X$, we have*

$$\langle x - y, T(x) - T(y)\rangle \le \langle x - y, \nabla h(x) - \nabla h(y)\rangle. \tag{15}$$

Also, T is called ρ −strictly h −pseudo-contractive maps if for each $x, y \in X$, we have

$$\langle x - y, T(x) - T(y)\rangle \le \langle x - y, \nabla h(x) - \nabla h(y)\rangle - \rho \parallel (\nabla h(x) - \nabla h(y)) - (T(x) - T(y)) \parallel^2. \tag{16}$$

Observe that h −fixed point problem of this mapping is to find a fixed point say $q \in K$ such that

$$Tq = \nabla h(q). \tag{17}$$

Note that T is h-pseudo-contractive $\Leftarrow$ $\nabla h - T$ is monotone, (see Zegeye et.al 2022) for more information).

Furthermore, system of generalized mixed equilibrium problems is defined as follows:

Definition 2.5 *For each $i = 1,2,\ldots,N$, we let K_i represent a non empty, closed and convex subset of X and $\cap_{i=1}^N K_i \neq \emptyset$. Let $\Theta_i: K_i \times K_i \to R$ be a bifunctions, $\theta_i: K_i \to R$ be a convex and lower semicontinuous functions, $A_i: K_i \to X^*$ be nonlinear mappings, for each $i = 1,2\ldots,N$. The system of generalized mixed equilibrium problems (SGMEP) is to find a point $z \in \cap_{i=1}^N K_i$ such that*

$$G_i(z, y) := \Theta_i(z, y) + \langle A_i z, y - z\rangle + \theta_i(y) + \theta_i(z) \ge 0, \forall y \in K_i, i = 1,2\ldots,N. \tag{18}$$

We observe for each $i = 1,2,\ldots,N$, (SGMEP)(18) is reduced to the generalized mixed equilibrium problems $(GMEP_i)$ which is to find $z \in K_i$ such that

$$G_i(z, y) := \Theta_i(z, y) + \langle A_i z, y - z\rangle + \theta_i(y) + \theta_i(z) \ge 0, \forall y \in K_i, i = 1,2\ldots,N. \tag{19}$$

The solution set of (??) is denoted by $GMEP_i(\Theta_i, A_i, \theta_i) = \Lambda_i$. Thus, the solution set of SGMEP(18) is denoted by $SGMEP(\Theta_i, A_i, \theta_i) = \cap_{i=1}^N GMEP_i(\Theta_i, A_i, \theta_i)$.

The bi-function Θ is said to satisfied **Condition A (A1)-(A4)** if the following conditions hold

1. $\Theta(x, x) = 0, \forall x \in K$,
2. $\Theta: K \times K \to \mathbb{R}$ is monotone, that is $\Theta(u, x) + \Theta(x, u) \le 0$,

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3. $\limsup_{t\downarrow 0}\Theta(\mu x+(1-\mu)u,y)\le\Theta(u,y),\forall x,y,u\in K$,
4. The function $x\mapsto\Theta(u,x)$ is convex and lower-semicontinuous.

Observe that the above equilibrium problem can be treated as a fixed point problem as follows. Given that $G^{h,r_n}: X\to K$ is a resolvent mapping corresponding to a bi-function Θ, defined for $r_n>0$ by $G^{h,r_n}(x)=\{z\in K: \Theta(z,y)+\frac{1}{r_n}\langle\nabla h(z)-\nabla h(x),y-z\rangle\ge 0,\forall y\in K\ \forall x\in X\}$, then by means of resolvent method, we have that x is a fixed point of G^{h,r_n} provided it is a solution of fixed point. These notion of equilibrium and generalized mixed equilibrium problems were originally introduced and studied by (Blum and Oettli, 1994, Mouda and Thera, 1999, Browder, 1966, Zegeye, 2022, Payvand & Jahedi 2016, Yousuf, 2019) and have been further studied by various authors cited in the literature.

Definition 2.6 *If h is strictly convex on $int(dom\ h)$ and K is a subset of X such that $int(dom\ h)\cap K\neq\emptyset$, then there exists at most one point $P_K^h(x)\in int(dom\ h)\cap K$ satisfying*

$$d_h(P_K^h(x),x)=inf\{d_h(z,x): z\in int(dom\ h)\cap K\}. \tag{20}$$

This point if exist is called the Bregman Projection of $x\in int(dom\ h)$ onto a nonempty closed and convex set K. For more information on important function properties, the reader may consult the following references (Reich & Sabach, 2010, Bregman, 1967, Censor & Lent, 1981, Phelps, 1993, Bauschke & Borwein, 1997, Bauschke et.al 2001, Butnariu & Resmerita, 2006, Butnariu & Iusem. 2000, Bonnas & Shapiro, 2000, Ekuma-Okereke & Oladipo, 2020, Alber, 1996 and host of others).

Main results

In this section, we introduce and solve the problems $[\cap_{i=1}^{\infty}Fix(T_t)]\cap[(\cap_{i=1}^{M}GMEP(\Theta_i,A_i,\theta_i))]\cap\left[(\cap_{j=1}^{N}Fix_h(S_j))\right]\neq\emptyset$ and prove that sequence of iterates strongly converges to the feasible set of these problems. To achieve this, we have the following results:

Lemma 3.1 *Assume that conditions $(H1)-(H9)$ hold, then **Algorithm 2.1** defined by (11) is well-defined.*

Proof. The proof severally separated in steps; **Step 1:** Show that $\Omega=[\cap_{i=1}^{C}Fix(T_t)]\cap[(\cap_{i=1}^{M}GMEP(\Theta_i,A_i,\theta_i))]\cap\left[(\cap_{j=1}^{N}Fix_h(S_j))\right]\neq\emptyset$ is closed and convex.

This is obviously true from a Lemma in Ugwunnadi et.al 2014, that $Fix(T_t)$ is closed and convex for any $t=1,2,\ldots,$. From Lemmas in Yongfu & Yongchun, 2015 and Zeeye, 2022, we get that $GMEP(\Theta_i,A_i,\theta_i)$ and $Fix_h(S_j)$ are closed and convex for any $i=1,2,\ldots,M$ and $j=1,2,\ldots,N$ respectively. Consequently, since the intersection of two or more convex set is convex, we get that Ω is closed and convex.

Step 2: Show that the set C_n is closed and convex for all $n\in\mathbb{N}$.

It is clear that $C_1=X$ is closed and convex for $n=1$. We assume that C_n is closed and convex for some $n\ge 1$ and $u\in C_{n+1}$. Set

$$Q_n=\{u\in X: d_h(u,y_n)\le d_h(u,z_n)+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla h(T_tz_n),z_n-u\rangle\}$$
$$A_n=\{u\in X: d_h(u,u_n)\le d_h(u,y_n)\}$$
$$B_n=\{u\in X: d_h(u,v_n)\le d_h(u,u_n)\},$$

such that $C_{n+1}=C_n\cap Q_n\cap A_n\cap B_n$. We need to show that Q_n, A_n, and B_n are closed and convex. First, we show that Q_n is closed and convex for $n\ge 1$.

$$Q_n=\{u\in X: d_h(u,y_n)\le d_h(u,z_n)+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla h(T_tz_n),z_n-u\rangle\}$$
$$=\{u\in X: d_h(u,y_n)-d_h(u,z_n)\le\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla h(T_tz_n),z_n-u\rangle\}$$
$$=\{u\in X: h(z_n)-h(y_n)-\langle\nabla(y_n),u-y_n\rangle+\langle\nabla(z_n),u-z_n\rangle\le\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla h(T_tz_n),z_n-u\rangle\}$$
$$=\{u\in X: \langle\nabla(z_n),u-z_n\rangle-\langle\nabla(y_n),u-y_n\rangle\le h(y_n)-h(z_n)+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla h(T_tz_n),z_n-u\rangle\}$$
$$=\{u\in X: \langle\nabla(z_n),u\rangle+\tfrac{\rho}{1-\rho}\langle\nabla(z_n),u\rangle-\langle\nabla(y_n),u\rangle-\tfrac{\rho}{1-\rho}\langle\nabla(Tz_n),u\rangle$$
$$\le h(y_n)-h(z_n)+\langle\nabla(z_n),z_n\rangle-\langle\nabla(y_n),y_n\rangle+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n),z_n\rangle-\tfrac{\rho}{1-\rho}\langle\nabla h(T_tz_n),z_n\rangle\}$$

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$= \{u \in X: \langle \frac{\rho}{1-\rho}\nabla(z_n) - \nabla(y_n) - \frac{\rho}{1-\rho}\nabla(Tz_n), u\rangle + \leq h(y_n) - h(z_n) + \langle \frac{\rho}{1-\rho}\nabla(z_n), z_n\rangle$
$-\langle \nabla(y_n), y_n\rangle - \langle \frac{\rho}{1-\rho}\nabla h(Tz_n), z_n\rangle\}.$

This shows that Q_n is closed and convex for all $n \geq 1$. Next, we show that A_n is closed and convex for all $n \geq 1$.

$A_n = \{u \in X: d_h(u, u_n) \leq d_h(u, y_n)\}$
$= \{u \in X: h(u_n) - h(y_n) + \langle \nabla h(u_n), u - u_n\} - \langle \nabla(y_n), u - y_n\rangle \leq 0\rangle$
$= \{u \in X: h(u_n) - h(y_n) + \langle \nabla h(y_n), y_n\rangle - \langle \nabla(u_n), u_n\rangle \leq \langle \nabla h(y_n) - \nabla(u_n), u\rangle\}.$

Similarly,

$B_n = \{u \in X: d_h(u, v_n) \leq d_h(u, u_n)\}$
$= \{u \in X: h(v_n) - h(u_n) + \langle \nabla h(v_n), u - u_n\} - \langle \nabla h(u_n), u - u_n\rangle \leq 0\rangle$
$= \{u \in X: h(v_n) - h(u_n) + \langle \nabla h(u_n), u_n\rangle - \langle \nabla h(v_n), v_n\rangle \leq \langle \nabla h(u_n) - \nabla(v_n), u\rangle\}.$

This shows that A_n and B_n are closed and convex for all $n \geq 1$. Hence, for all $n \geq 1$, C_n is closed and convex.

Step 3: Show that $\Omega \subset C_n$, for all $n \in \mathbb{N}$.

Let $\delta_0 = \gamma_0 = I, \delta_j = R^{h,r_n}_{S_j} \circ R^{h,r_n}_{S_{j-1}} \circ \ldots R^{h,r_n}_{S_2} \circ R^{h,r_n}_{S_1}$, for $j = 1,2, \ldots N$ and $\gamma_i = G^{h,r_n}_{H_i} \circ G^{h,r_n}_{H_{i-1}} \ldots \circ G^{h,r_n}_{H_2} \circ G^{h,r_n}_{H_1}$, for $i = 1,2, \ldots M$. Let $q \in \Omega$, then by invoking Lemmas in Yongfu & Yongchun, 2015 and Zeeye, 2022, Ugwunnadi, 2014, we compute as follows:

$d_h(q, v_n) = d_h(q, \delta_j u_n) \leq d_h(q, \delta_{N-1}u_n) - d_h(u_n, \delta_{N-1}u_n)$
$\leq d_h(q, \delta_{N-2}u_n) - d_h(\delta_{N-1}u_n, \delta_{N-2}u_n) - d_h(u_n, \delta_{N-1}u_n)$
$\vdots$
$\leq d_h(q, \delta_0 u_n) - d_h(\delta_1 u_n, \delta_0 u_n) - d_h(u_n, \delta_1 u_n).$

By induction, we get

$$d_h(q, v_n) \leq d_h(q, u_n) - \sum_{j=0}^{N-1} d_h(\delta_{j+1}u_n, \delta_j u_n). \quad (21)$$

Continuing the process

$d_h(q, u_n) \leq d_h(q, \gamma_{M-1}y_n) - d_h(y_n, \gamma_{M-1}y_n)$
$\leq d_h(q, \gamma_{M-2}y_n) - d_h(\gamma_{M-1}y_n, \gamma_{M-2}y_n) - d_h(y_n, \gamma_{M-1}y_n)$
$\vdots$
$\leq d_h(q, \gamma_0 y_n) - d_h(\gamma_1 y_n, \gamma_0 y_n) - d_h(y_n, \gamma_1 y_n).$

Again by induction, we get

$$d_h(q, u_n) \leq d_h(q, y_n) - \sum_{i=0}^{M-1} d_h(\gamma_{i+1}y_n, \gamma_i y_n). \quad (22)$$

In addition, we get

$d_h(q, y_n) = d_h(q, \nabla h^*(b_n \nabla h(z_n) + (1 - b_n)\nabla h(T_t z_n)))$
$\leq d_h(q, z_n) + \rho d_h(z_n, Tz_n)$

$$\leq d_h(q, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla(Tz_n), z_n - q\rangle. \quad (23)$$

Combining inequalities (21), (22) and (23) together with Lemma in Wega & Zegeye, 2020, we get

$$d_h(q, v_n) \leq d_h(q, u_n) \leq d_h(q, y_n) \leq d_h(q, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla(Tz_n), z_n - q\rangle$$
$$- \sum_{j=0}^{N-1} d_h(\delta_{j+1}u_n, \delta_j u_n) - \sum_{i=0}^{M-1} d_h(\gamma_{i+1}y_n, \gamma_i y_n)$$
$$d_h(q, v_n) \leq d_h(q, u_n) \leq d_h(q, y_n) \leq d_h(q, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla(Tz_n), z_n - q\rangle$$
$$-\frac{\mu}{2}\left(\sum_{j=0}^{N-1} \| \delta_{j+1}u_n - \delta_j u_n \|^2 + \sum_{i=0}^{M-1} \| \gamma_{i+1}y_n - \gamma_i y_n \|^2\right) \quad (24)$$
$$d_h(q, v_n) \leq d_h(q, u_n) \leq d_h(q, y_n) \leq d_h(q, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla(Tz_n), z_n - q\rangle. \quad (25)$$

Hence $q \in C_{n+1}$ and consequently, $\Omega \in C_{n+1}$ for $n \geq 1$. Thus, $\Omega \subset C_n$. This completes the proof that **Algorithm 2.1** defined by (11) is well-defined.

Theorem 3.2 *Assume that conditions $(H1) - (H9)$ hold, then the sequences $\{x_n\}$, $\{z_n\}$ generated by **Algorithm 2.1** converges strongly to $q = P^h_\Omega(x_0)$ nearest to x_0, as $n \to \infty$ and where P^h_Ω is the Bregman projection mapping of C_{n+1} onto Ω.*

Proof. The proof is as follows. **Step 4:** Show that the sequences are bounded.

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To justify, notice in (11), that $x_n = P^h_{C_n}(x_0)$ and $x_{n+1} = P^h_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$. Thus, using part b of the Lemma in Reich & Sabach, 2010, we have that

$$d_h(x_n, x_0) \le d_h(x_{n+1}, x_0) - d_h(x_{n+1}, x_n)$$
$$d_h(x_{n+1}, x_0) \ge d_h(x_n, x_0). \quad (26)$$

This shows that the sequence $\{d_h(x_n, x_0)\}$ is increasing. Again, we have $\forall n \in \mathbb{N}, q \in \Omega$ that

$$d_h(x_n, x_0) = d_h(P^h_{C_n}(x_0), x_0)$$
$$\le d_h(q, x_0) - d_h(q, P^h_{C_n}(x_0))$$
$$\le d_h(q, x_0). \quad (27)$$

This proves that $\{d_h(x_n, x_0)\}$ is bounded. From Lemma in Butnariu & Iusem, 2000, $\{x_n\}$ is bounded. Combining (26) and (27) proves that $\lim_{n\to\infty} d_h(x_n, x_0)$ exist. Now, without loss of generality, let

$$\lim_{n\to\infty} d_h(x_n, x_0) = l. \quad (28)$$

In addition to (28), we get for any positive integer k and as $n \to \infty$, that

$$d_h(x_{n+k}, x_n) = d_h(x_{n+k}, P^h_{C_n}(x_0))$$
$$\le d_h(x_{n+k}, x_0) - d_h(x_n, x_0) \to 0. \quad (29)$$

So that $\lim_{k\to\infty} d_h(x_{n+k}, x_n) = 0$. In particular,

$$\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0. \quad (30)$$

Thus we obtain using Lemma in Butnariu & Iusem, 2000, that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \quad (31)$$

This implies that $\{x_n\}$ is a Cauchy sequence. Now since ∇h is bounded and uniformly norm-to-norm continuous on bounded subsets of X by the Lemmas of . Naraghirad & Yao, 2013, Zegeye & Shahzad, 2014, we obtain that

$$\lim_{n\to\infty} ||\nabla h(x_{n+1}) - \nabla h(x_n)|| = 0. \quad (32)$$

Furthermore, we obtain from the definition of z_n and together with (31) as $n \to \infty$, that

$$||x_n - z_n|| = ||x_n - x_n - \alpha_n(x_n - x_{n-1})||$$
$$= ||\alpha_n(x_{n-1} - x_n)||$$
$$\le ||x_{n-1} - x_n|| \to 0.$$

We get that

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \quad (33)$$

Consequently, we obtain that

$$\lim_{n\to\infty} ||\nabla h(x_n) - \nabla h(z_n)|| = 0. \quad (34)$$

By (33), z_n is bounded. Consequent upon the boundedness of ∇h and by (34), $\nabla h(z_n)$ and $\nabla h(x_n)$ are bounded. Moreover, since x_n, z_n are bounded, together with (33) and (34), we get that

$$\lim_{n\to\infty} d_h(x_n, z_n) = -\lim_{n\to\infty} d_h(z_n, x_n) + \lim_{n\to\infty} \langle x_n - z_n, \nabla h(x_n) - \nabla h(z_n)\rangle$$
$$\le \lim_{n\to\infty} \| x_n - z_n \| . \lim_{n\to\infty} \| \nabla h(x_n) - \nabla h(z_n) \| = 0. \quad (35)$$

Thus

$$\lim_{n\to\infty} d_h(x_n, z_n) = 0. \quad (36)$$

Consequently, since ∇h is bounded, we have

$$d_h(x_{n+1}, z_n) = d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + \langle \nabla h(x_n) - \nabla h(z_n), x_{n+1} - x_n\rangle$$
$$\le d_h(x_{n+1}, x_n) + d_h(x_n, z_n) + ||\nabla h(x_n) - \nabla h(z_n)||. ||x_{n+1} - x_n||.$$

Using (30),(31), (34) and (36) we get

$$\lim_{n\to\infty} d_h(x_{n+1}, z_n) = 0. \quad (37)$$

Using Lemma in [32], we obtain as $n \to \infty$ that

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0. \quad (38)$$

Since from (11), $x_{n+1} \in C_{n+1} \subset C_n$,

$$d_h(x_{n+1}, v_n) \le d_h(x_{n+1}, u_n) \le d_h(x_{n+1}, y_n) \le d_h(x_{n+1}, z_n) + \frac{\rho}{1-\rho}\langle \nabla h(z_n) - \nabla(Tz_n), z_n - x_{n+1}\rangle. \quad (39)$$

It follows from (37),(38) and the boundedness of ∇h that (39) becomes

$$\lim_{n\to\infty} d_h(x_{n+1}, v_n) = 0 \quad (40)$$

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Thus, by the Lemma in Butnariu and Iusem, 2000, we obtain that since from the hypothesis, h is totally convex on bounded subset of X, h is sequentially consistent.So it follows that

$$\lim_{n\to\infty}||x_{n+1}-v_n||=0. \quad (41)$$

Besides,

$$\left||x_n-v_n|\right|\leq\left||x_n-x_{n+1}|\right|+\left||x_{n+1}-v_n|\right| \; implies \lim_{n\to\infty}||x_n-v_n||=0, \quad (42)$$

and

$$\left||v_n-z_n|\right|\leq\left||v_n-x_n|\right|+\left||x_n-z_n|\right| \text{ implies } \lim_{n\to\infty}||v_n-z_n||=0. \quad (43)$$

This demonstrates that $\{v_n\}$ is bounded. Now since $\nabla(h)$ is norm-to-norm uniformly continuous on bounded subsets of X, we get

$$\lim_{n\to\infty}||\nabla h(v_n)-\nabla h(z_n)||=0. \quad (44)$$

Now since $v_n=\delta_j u_n$, we have for any $q\in\Omega$ that

$$\begin{aligned}
&d_h(q,v_n)\leq d_h(q,z_n)+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla(T_t z_n),z_n-q\rangle\\
&-\tfrac{\mu}{2}\left(\textstyle\sum_{j=0}^{N-1}\parallel\delta_{j+1}u_n-\delta_j u_n\parallel^2+\sum_{i=0}^{M-1}\parallel\gamma_{i+1}y_n-\gamma_i y_n\parallel^2\right)\\
&\leq d_h(q,z_n)-d_h(q,v_n)+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla(T_t z_n),z_n-q\rangle\\
&=h(v_n)-h(z_n)-\langle\nabla h(z_n)-\nabla h(v_n),v_n-z_n\rangle\\
&+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla(T_t z_n),z_n-q\rangle\\
&=d_h(v_n,z_n)+\langle\nabla h(v_n),v_n-z_n\rangle\\
&+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla(T_t z_n),z_n-q\rangle\\
&=-d_h(z_n,v_n)+\langle v_n-z_n,\nabla h(v_n)-\nabla h(z_n)\rangle\\
&+\langle\nabla h(v_n),v_n-z_n\rangle+\tfrac{\rho}{1-\rho}\langle\nabla h(z_n)-\nabla(T_t z_n),z_n-q\rangle\\
&\leq\parallel v_n-z_n\parallel\parallel\nabla h(v_n)-\nabla h(z_n)\parallel+\parallel\nabla h(v_n)\parallel\parallel v_n-z_n\parallel\\
&+\tfrac{\rho}{1-\rho}\parallel\nabla h(z_n)-\nabla(T_t z_n)\parallel\parallel z_n-q\parallel
\end{aligned} \quad (45)$$

Invoking (43), (44) and for any $q\in\Omega$ in inequality (45), we have that

$$\lim_{n\to\infty}\left(\textstyle\sum_{j=0}^{N-1}\parallel\delta_{j+1}u_n-\delta_j u_n\parallel^2+\sum_{i=0}^{M-1}\parallel\gamma_{i+1}y_n-\gamma_i y_n\parallel^2\right)=0. \quad (46)$$

This implies that

$$\lim_{n\to\infty}\parallel\delta_{j+1}u_n-\delta_j u_n\parallel=0, 0\leq j\leq N-1, \quad (47)$$

and hence

$$\lim_{n\to\infty}\parallel v_n-u_n\parallel=0. \quad (48)$$

Since $\nabla(h)$ is norm-to-norm uniformly continuous on bounded subsets of X, we get

$$\lim_{n\to\infty}\parallel\nabla h(v_n)-\nabla h(u_n)\parallel=0. \quad (49)$$

In view of these, we get

$$\begin{aligned}
&\lim_{n\to\infty}d_h(v_n,u_n)=-\lim_{n\to\infty}d_h(u_n,v_n)+\lim_{n\to\infty}\langle v_n-u_n,\nabla h(v_n)-\nabla h(u_n)\rangle\\
&\leq\lim_{n\to\infty}\parallel v_n-u_n\parallel.\lim_{n\to\infty}\parallel\nabla h(v_n)-\nabla h(u_n)\parallel=0.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}d_h(v_n,u_n)=0 \quad (50)$$

Similarly,

$$\lim_{n\to\infty}\parallel\gamma_{i+1}u_n-\gamma_i u_n\parallel=0, 0\leq i\leq M-1, \quad (51)$$

and hence

$$\lim_{n\to\infty}\parallel u_n-y_n\parallel=0. \quad (52)$$

Since $\nabla(h)$ is norm-to-norm uniformly continuous on bounded subsets of X, we get

$$\lim_{n\to\infty}\parallel\nabla h(u_n)-\nabla h(y_n)\parallel=0. \quad (53)$$

In view of these, we get

$$\begin{aligned}
&\lim_{n\to\infty}d_h(u_n,y_n)==-\lim_{n\to\infty}d_h(y_n,u_n)+\lim_{n\to\infty}\langle u_n-y_n,\nabla h(u_n)-\nabla h(y_n)\rangle\\
&\leq\lim_{n\to\infty}\parallel u_n-v_n\parallel.\lim_{n\to\infty}\parallel\nabla h(u_n)-\nabla h(y_n)\parallel=0.,
\end{aligned}$$

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Thus
$$\lim_{n\to\infty} d_h(u_n, y_n) = 0 \tag{54}$$
Besides, we have from (44) and (49) that
$$\lim_{n\to\infty} \| \nabla h(u_n) - \nabla h(z_n) \| \leq \lim_{n\to\infty} \| \nabla h(u_n) - \nabla h(v_n) \| + \lim_{n\to\infty} \| \nabla h(v_n) - \nabla h(z_n) \| = 0$$
Thus
$$\lim_{n\to\infty} \| \nabla h(u_n) - \nabla h(z_n) \| = 0. \tag{55}$$
Consequently, we have from (53) and (55) that
$$\lim_{n\to\infty} \| \nabla h(y_n) - \nabla h(z_n) \| \leq \lim_{n\to\infty} \| \nabla h(y_n) - \nabla h(u_n) \| + \lim_{n\to\infty} \| \nabla h(u_n) - \nabla h(z_n) \| = 0$$
Thus
$$\lim_{n\to\infty} \| \nabla h(y_n) - \nabla h(z_n) \| = 0. \tag{56}$$
This demonstrates that $\{v_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded. Now since $\nabla(h)$ is norm-to-norm uniformly continuous on bounded subsets of X we also get that $\{\nabla h(u_n)\}$, $\{\nabla h(v_n)\}$ and $\{\nabla h(y_n)\}$ are bounded. Hence, with (56), we get from $y_n = \nabla h^*\big(b_n \nabla h(z_n) + (1-b_n)\nabla h(T_t z_n)\big)$, that
$$\| \nabla h(y_n) - \nabla h(z_n) \| = (1-b_n) \| \nabla h(T_t z_n) - \nabla h(z_n) \|$$
$$\| \nabla h(T_t z_n) - \nabla h(z_n) \| = \frac{1}{(1-b_n)} \| \nabla h(y_n) - \nabla h(z_n) \|$$

$$\lim_{n\to\infty} \| \nabla h(T_t z_n) - \nabla h(z_n) \| = 0. \tag{57}$$
But ∇h^* is norm-to-norm uniformly continuous on bounded subsets of X^*, so we get
$$\lim_{n\to\infty} \| T_t z_n - z_n \| = 0 \tag{58}$$
Step 5: Show that $q \in \Omega := [\cap_{i=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} GMEP(\Theta_i, A_i, \theta_i))] \cap \big[(\cap_{j=1}^{N} Fix_h(S_j))\big]$

First, we show that $q \in \cap_{t=1}^{\infty} Fix(T_t)$ for each $t \geq 1$. Using the fact that X is reflexive, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \hookrightarrow q$ as $k \to \infty$. From (58),and the fact that $(I - T_t)$ for each $t \geq 1$ is demi-closed, we have that $q = T_t q$ for any $t \geq 1$. Thus, this shows that $q \in Fix(T_t)$ and hence $q \in \cap_{t=1}^{\infty} Fix(T_t)$. In view of the fact that from (33), $\{z_n\}$ is a Cauchy sequence, we have that $z_{n_k} \to q$.

Next we show that $q \in \cap_{i=1}^{M} GMEP(\Theta_i, A_i, \theta_i)$. Set $\gamma_i(y_{n_k}) = G_{H_i}^{h, r_{n_k}} \gamma_{i-1}(y_{n_k})$ and $H_i(\gamma_i(y_{n_k}), y) = \Theta_i(\gamma_i(y_{n_k}), y) + \langle y - \gamma_i(y_{n_k}), A_i z\rangle + \theta_i(y) - \theta_i(\gamma_i(y_{n_k}))$, then
$$H_i(\gamma_i(y_{n_k}), y) + \frac{1}{r_{n_k}} \langle y - \gamma_i(y_{n_k}), \nabla h(\gamma_i(y_{n_k})) - \nabla h(\gamma_{i-1}(y_{n_k}))\rangle \geq 0, \forall y \in K.$$
Applying **Condition A (A2)** we have
$$H_i(y, \gamma_i(y_{n_k})) \leq -H_i(\gamma_i(y_{n_k}), y) \leq \frac{1}{r_{n_k}} \langle y - \gamma_i(y_{n_k}), \nabla h(\gamma_i(y_{n_k})) - \nabla h(\gamma_{i-1}(y_{n_k}))\rangle$$
$$\leq \| y - \gamma_i(y_{n_k}) \| \frac{\|\nabla h(\gamma_i(y_{n_k})) - \nabla h(\gamma_{i-1}(y_{n_k}))\|}{r_{n_k}}$$
$$\leq \max_{1\leq i\leq M} \sup_{k\geq 0} \{\| y - \gamma_i(y_{n_k}) \|\} \frac{\|\nabla h(\gamma_i(y_{n_k})) - \nabla h(\gamma_{i-1}(y_{n_k}))\|}{r_{n_k}}. \tag{59}$$
Using the fact $\gamma_i(y_{n_k}) \to q$, the fact that $H(y, .)$ is convex and lower semi-continuous **(Condition A (A4))**, $r_{n_k} \geq c$ and taking limits of both sides of (59), we get that
$$H_i(y, q) \leq 0, \forall y \in K. \tag{60}$$
Set $y_\alpha = \alpha y + (1-\alpha) q$ and this suggest that $y_\alpha \in K$. This implies from (60) that $H_i(y_\alpha, q) \leq 0$. It follows from this and **Condition A** (**A1**) and (**A4**) that
$$0 = H_i(y_\alpha, y_\alpha) = H_i(y_\alpha, \alpha y + (1-\alpha) q)$$
$$\leq \alpha H_i(y_\alpha, y) + (1-\alpha) H_i(y_\alpha, q)$$
$$\leq \alpha H_i(y_\alpha, y)$$
$$\leq H_i(y_\alpha, y).$$
Thus
$$0 \leq H_i(y_\alpha, y). \tag{61}$$
Using (61) and **Condition A** (**A3**), we have that
$$H_i(q, y) \geq 0. \tag{62}$$
Thus $q \in \cap_{i=1}^{M} GMEP(\Theta_i, A_i, \theta_i)$

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Furthermore, we show that $q \in \cap_{j=1}^{N} Fix_h(S_j)$. Let $\delta_j(u_{n_k}) = R_{S_j}^{h,r_{n_k}} \delta_{j-1}(u_{n_k})$. Invoking a Lemma in Yongfu & Yongchun, 2015, we obtain

$$\langle y - \delta_j(u_{n_k}), S_j\delta_j(u_{n_k})\rangle - \frac{1}{r_{n_k}}\langle y - \delta_j(u_{n_k}), (1 + r_{n_k})\nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k}))\rangle \leq 0.$$

By the convexity of K, we set $y_\alpha = \alpha y + (1 - \alpha)q \in K$, for $y \in K, \alpha \in [0,1]$. Thus,

$$\begin{aligned}
&\langle \delta_j(u_{n_k}) - y_\alpha, S_j y_\alpha\rangle \geq \langle \delta_j(u_{n_k}) - y_\alpha, S_j y_\alpha\rangle + \langle y_\alpha - \delta_j(u_{n_k}), S_j\delta_j(u_{n_k})\rangle \\
&-\frac{1}{r_{n_k}}\langle y_\alpha - \delta_j(u_{n_k}), (1 + r_{n_k})\nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k}))\rangle \\
&= \langle \delta_j(u_{n_k}) - y_\alpha, S_j y_\alpha - S_j\delta_j(u_{n_k})\rangle \\
&-\frac{1}{r_{n_k}}\langle y_\alpha - \delta_j(u_{n_k}), (1 + r_{n_k})\nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k}))\rangle \\
&\geq \langle \delta_j(u_{n_k}) - y_\alpha, \nabla h(y_\alpha) - \nabla h(\delta_j(u_{n_k}))\rangle \\
&-\frac{1}{r_{n_k}}\langle y_\alpha - \delta_j(u_{n_k}), (1 + r_{n_k})\nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k}))\rangle \\
&= \langle \delta_j(u_{n_k}) - y_\alpha, \nabla h(y_\alpha)\rangle \\
&-\frac{1}{r_{n_k}}\langle y_\alpha - \delta_j(u_{n_k}), \nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k}))\rangle \\
&\geq \langle \delta_j(u_{n_k}) - y_\alpha, \nabla h(y_\alpha)\rangle \\
&-\| y_\alpha - \delta_j(u_{n_k}) \| \| \nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k})) \| \frac{1}{r_{n_k}} \\
&\geq \langle \delta_j(u_{n_k}) - y_\alpha, \nabla h(y_\alpha)\rangle \\
&- \max_{1\leq i\leq M} \sup_{k\geq 0}\{\| y_\alpha - \delta_j(u_{n_k}) \|\rangle \| \nabla h(\delta_j(u_{n_k})) - \nabla h(\delta_{j-1}(u_{n_k})) \| \frac{1}{r_{n_k}}
\end{aligned}$$

Using the fact $\delta_j(u_{n_k}) \to q$, the fact that ∇h is uniformly continuous, $r_{n_k} \geq c$ and taking limits of both sides of (??), we get that

$$\langle q - y_\alpha, S_j y_\alpha\rangle \geq \langle q - y_\alpha, \nabla h(y_\alpha)\rangle.$$

This implies

$$\langle q - y, S_j(q + \alpha(y - q))\rangle \geq \langle q - y, \nabla h(q + \alpha(y - q))\rangle. \tag{63}$$

Since S_j is continuous for each j and the fact that ∇h is uniformly continuous on bounded subset of X, and by letting $\alpha \downarrow 0$, we get that

$$\langle q - y, S_j(q)\rangle \geq \langle q - y, \nabla h(q)\rangle \tag{64}$$

$$\equiv \langle q - y, \nabla h(q) - S_j(q)\rangle \leq 0. \tag{65}$$

Setting $y = \nabla h^*(S_j q)$, we have

$$\langle q - \nabla h^*(S_j q), \nabla h(q) - \nabla h^*(\nabla h(S_j(q)))\rangle \leq 0 \tag{66}$$

Since ∇h^* is monotone, we have that

$$\langle q - \nabla h^*(S_j q), \nabla h(q) - \nabla h^*(\nabla h(S_j(q)))\rangle = 0 \tag{67}$$

This implies that

$$\nabla h^*(\nabla h(S_j(q)) = \nabla h(q). \tag{68}$$

Hence, $q \in Fix_h(S_j)$, for each $j = 1,2, \dots N$ and $q \in \cap_{j=1}^{N} Fix_h(S_j)$

Let $\delta_0 = \gamma_0 = I, \delta_j = R_{S_j}^{h,r_n} \circ R_{S_{j-1}}^{h,r_n} \circ \dots R_{S_2}^{h,r_n} \circ R_{S_1}^{h,r_n}$, for $j = 1,2, \dots N$ and

Step 6: Show that $q = P_\Omega^h(x_0)$.

Set $u = P_\Omega^h(x_0)$. In Lemma 3.1, we demonstrated that $\Omega \subset C_n$. Since $P_\Omega^h(x_0) \in \Omega$, we get that $P_\Omega^h(x_0) \subset C_n$. It then follows from our setting with $x_n = P_{C_n}^h(x_0)$ that

$$d_h(x_n, x_0) \leq d_h(u, x_0). \tag{69}$$

Since $x_n \to q$ as $n \to \infty$, we get from (69) that

$$d_h(q, x_0) \leq d_h(u, x_0). \tag{70}$$

But by the Property of Bregman Projection mapping in Lemma of Reich & Sabach, 2010, we have for $\forall w \in \Omega$ that

$$d_h(u, x_0) \leq d_h(w, x_0). \tag{71}$$

This implies

$$d_h(u, x_0) \leq d_h(q, x_0). \tag{72}$$

Thus by combining (70) and (72) we then arrive at $u = q$. Therefore, $q = P_\Omega^h(x_0)$. This completes the proof of our Theorem 3.2.

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Application

System of equilibrium problems, finite family of h-pseudo-contractive mappings and countable families of Bregman quasi strictly pseudo-contractive mappings

By setting $A_i \equiv 0$ and $\theta_i \equiv 0$ in our Hypothesis 2.1, then the sequence x_n and z_n defined and generated by **Algorithm 2.1** (11) converges to the common solution set denoted by

$$\Omega := [\cap_{i=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} EP(\Theta_i))] \cap \left[(\cap_{j=1}^{N} Fix_h(S_j))\right] \neq \emptyset;$$

where $EP(\Theta)$ is the set of solution of equilibrium problem of Θ.

System of Convex Minimization Problems, finite family of h-pseudo-contractive mappings and countable families of Bregman quasi strictly pseudo-contractive mappings

By setting $A_i \equiv 0$ and $\Theta_i \equiv 0$ in our Hypothesis 2.1, then the sequence x_n and z_n defined and generated by **Algorithm 2.1** (11) converges to the common solution set denoted by

$$\Omega := [\cap_{i=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} CMP(\theta_i))] \cap \left[(\cap_{j=1}^{N} Fix_h(S_j))\right] \neq \emptyset;$$

where $CMP(\theta)$ is the set of solution of convex minimization problem of θ.

System of Variational Inequality Problems, finite family of h-pseudo-contractive mappings and countable families of Bregman quasi strictly pseudo-contractive mappings

By setting $\Theta_i \equiv 0$ and $\theta_i \equiv 0$ in our Hypothesis 2.1, then the sequence x_n and z_n defined and generated by **Algorithm 2.1** (11) converges to the common solution set denoted by

$$\Omega := [\cap_{t=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} VIP(A_i, K))] \cap \left[(\cap_{j=1}^{N} Fix_h(S_j))\right] \neq \emptyset;$$

where $VIP(A_i, K)$ is the set of solution of variational inequality problem of A over K.

Numerical example

Let $X = \mathbb{R}$, $K = [-1,1]$, $h(x) := \frac{1}{2}x^2$, $\forall x \in K$, then $\nabla h = J = I, \nabla h^* = J = I$, where I is identity mapping on X. Let $\Theta_i(x,y) = \frac{i}{i+1}(y^2 - yx + xy - x^2), \forall y \in K, i = 1, \ldots M$. It is very obvious that $\Theta(x,y)$ satisfies **Condition A** $A1 - A4$ given in **Assumption 1**. Also let $\langle y - x, A_i x\rangle := (i+1)(yx - x^2)$ such that $A_i(x) := (i+1)x$. Let $\theta_i(x) = \theta_i(y) =$ constant. Let $T_t(x) = -x^2 - x, t = 1, \ldots$ and $S_j(x) = -s^j \nabla h(x), j = 1, \ldots, N, s = [0,1]$ be Bregman quasi-strict pseudo-contractive and h-pseudo-contraction mappings respectively such that

$$\Omega := [\cap_{t=1}^{\infty} Fix(T_t)] \cap [(\cap_{i=1}^{M} GMEP(\Theta_i, A_i, \theta_i))] \cap \left[(\cap_{j=1}^{N} Fix_h(S_j))\right] = \{0\}$$

Thus, for implementation of the algorithm, we set $\rho = 0.3; x_0 = x_1 = 1, S = 0.5, a_n = 0.3, b_n = 0.8, r = 1, M = N = 1$

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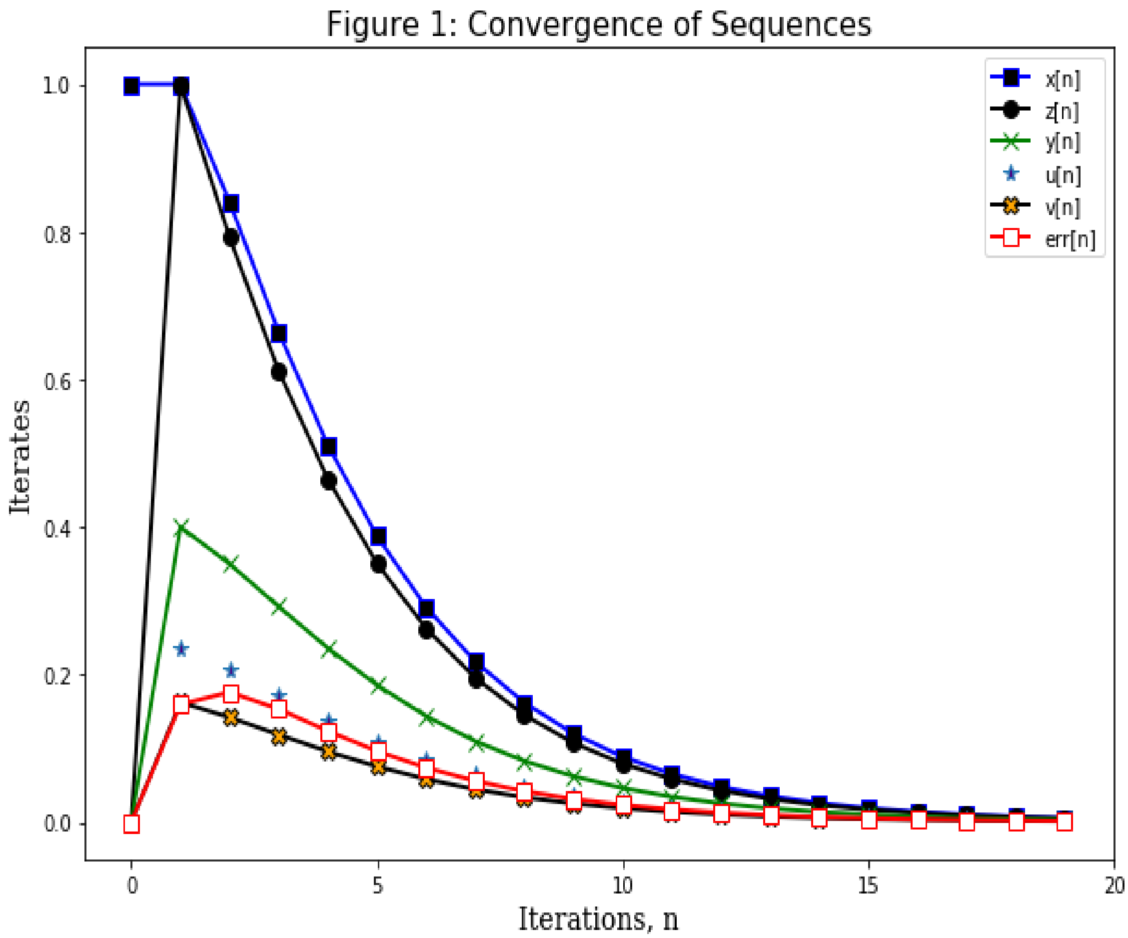

Conclusion

In this paper, we constructed a new hybrid algorithm to approximate the common solution of the set of fixed points of a countable family of closed Bregman quasi-strict pseudocontractive mappings; set of solutions of a finite system of a generalized mixed equilibrium problems and the set of f-fixed points of a finite family of f-pseudocontractive mappings. A strong convergence theorem was proved for it in a reflexive (real) Banach space. We provided some applications. Besides, a numerical computation was provided to illustrate its effectiveness and implementation. We observed that our results complement the results of Zegeye et.al 2022 and improves the results of Matsushita & Takahashi, 2005, Chen, 2011, Ceng & Yao, 2008, Yongfu & Yongchun, 2015, Wang & Wei, 2017, Wang, 2015 and many other references cited in the literature.

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