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Grid-Based and Hybrid Five-Step Block Runge-Kutta Methods for Solving Linear and Nonlinear First-Order Ordinary Differential Equations

[1]Adams, N.O., [2]Badmus, A.M., & [1]Yahuza, M.
[1]Department of Mathematics, Federal University of Education Zaria, Kaduna State, Nigeria.
[2]Department of Mathematical Sciences, Nigerian Defence Academy, Kaduna State Nigeria

***Corresponding author email**: orahanoadams@gmail.com

Abstract
The development of grid based and hybrid 5^{th} Step Block Runge–Kutta were constructed for the solution of both linear and non-linear first order ordinary differential equations through interpolation and collocation approach. The 5^{th} step Hybrid block Runge–Kutta method converges faster with two problems tested.

Keywords: Grid Based, Hybrid, Block R-K Method, linear and nonlinear.

Introduction
The Runge-Kutta Method is one of the numerical techniques used to approximate solutions of Ordinary Differential Equations (ODEs). It has become a popular solver for ODEs due to its effectiveness. This method has proven to be a promising approach for tackling both linear and nonlinear ODEs, employing various structures to enhance accuracy, stability, and efficiency.

Mshelia et al. (2016) introduced a hybrid block method for k=2 and later adapted it into a Fifth Stage Implicit R-K method for solving initial value problems of first-order differential equations. Ogunniran et al. (2020) evaluated the efficiency of Runge-Kutta methods in computational differential equations, focusing on the linear stability and capabilities of selected explicit R-K methods for integrating singular Lane-Emden differential equations. Kedir (2021) confirmed that a classical fourth-order Runge-Kutta method is suitable for solving first-order ordinary differential equations. Abualnaja (2015) developed a block procedure for K-step linear multi-step methods at K=1,2,3 using Legendre polynomials as the basis function. These discrete methods were implemented in block form to solve non-stiff initial value problems. This idea will be adopted to form block Runge-Kutta methods for both grid and hybrid based for solution of linear and nonlinear ODEs

The new Block Runge-Kutta Type Method offers several advantages, including cost-effectiveness, stability, rapid convergence, and suitability for handling both linear and nonlinear models.

Aim and Objectives of the Study
The study aims to adopt some Block Runge-Kutta methods for solution of linear and non-linear problems of first order ordinary differential equations. The objectives are to:

i. Adopt some K-step implicit linear multistep methods
ii. Reformulate the discrete schemes obtained in (i) into Block Runge-Kutta method
iii. Test the performance of the method by using problem of linear and non-linear first order ODEs.

Methodology

Derivation of some k-step Linear Multistep Methods

Consider a power series of variable x of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^j \quad (1)$$

as the approximate solution to equation $y' = f(x, y)$

Now the derivative of

$$y(x) = \sum_{j=0}^{m+t-1} a_j x^j \quad (2)$$

$$y'(x) = \sum_{j=1}^{m+t-1} j a_j x^{j-1} \quad (3)$$

where a_j is the parameters to be determined, m and t are numbers of the interpolation and collocation points respectively. Also, the degree of approximate solution as $(m + t - 1)$.

Derivation of Grid 5^{th} Step Linear Multistep Method

From equation (2) $m = 6, t = 1$, we have

$$y(x) = \sum_{j=0}^{6} a_j x^j \quad (4)$$

The collocation equation is

$$y'(x) = \sum_{j=1}^{6} j a_j x^{j-1} \quad (5)$$

Interpolating equation (4) at $x = x_n$ and collocating equation (5) at $x = x_{n+j}$, where $j = 0, 1, 2, 3, 4, 5$ gives the following system of equations

$$\begin{aligned}
y(x_n) &= a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 \\
f(x_n) &= a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 + 6a_6 x_n^5 \\
f(x_{n+1}) &= a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 + 6a_6 x_{n+1}^5 \\
f(x_{n+2}) &= a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + 4a_4 x_{n+2}^3 + 5a_5 x_{n+2}^4 + 6a_6 x_{n+2}^5 \\
f(x_{n+3}) &= a_1 + 2a_2 x_{n+3} + 3a_3 x_{n+3}^2 + 4a_4 x_{n+3}^3 + 5a_5 x_{n+3}^4 + 6a_6 x_{n+3}^5 \\
f(x_{n+4}) &= a_1 + 2a_2 x_{n+4} + 3a_3 x_{n+4}^2 + 4a_4 x_{n+4}^3 + 5a_5 x_{n+4}^4 + 6a_6 x_{n+4}^5 \\
f(x_{n+5}) &= a_1 + 2a_2 x_{n+5} + 3a_3 x_{n+5}^2 + 4a_4 x_{n+5}^3 + 5a_5 x_{n+5}^4 + 6a_6 x_{n+5}^5 \quad (6)
\end{aligned}$$

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Arranging equation (6) in matrix form

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \quad (7)$$

Determining the unknown parameters a_j where $j = 0,1,2,3,4,5,6$ and substituting them into equation (4) gives

$$y(x) = y_n + \left[x - \frac{137}{120}\frac{x^2}{h} + \frac{5}{8}\frac{x^3}{h^2} - \frac{17}{96}\frac{x^4}{h^3} + \frac{1}{40}\frac{x^5}{h^4} - \frac{1}{720}\frac{x^6}{h^5}\right] f_n + \left[\frac{5}{2}\frac{x^2}{h} - \frac{77}{36}\frac{x^3}{h^2} + \frac{71}{96}\frac{x^4}{h^3} - \frac{7}{60}\frac{x^5}{h^4} + \frac{1}{144}\frac{x^6}{h^5}\right] f_{n+1} + \left[\frac{5}{2}\frac{x^2}{h} + \frac{107}{36}\frac{x^3}{h^2} - \frac{59}{48}\frac{x^4}{h^3} + \frac{13}{60}\frac{x^5}{h^4} - \frac{1}{72}\frac{x^6}{h^5}\right] f_{n+2} + \left[\frac{5}{3}\frac{x^2}{h} - \frac{13x^3}{6h^2} + \frac{49}{48}\frac{x^4}{h^3} - \frac{1}{5}\frac{x^5}{h^4} + \frac{1}{72}\frac{x^6}{h^5}\right] f_{n+3} + \left[-\frac{5}{8}\frac{x^2}{h} + \frac{61}{71}\frac{x^3}{h^2} - \frac{41}{96}\frac{x^4}{h^3} + \frac{11}{120}\frac{x^5}{h^4} - \frac{1}{144}\frac{x^6}{h^5}\right] f_{n+4} + \left[\frac{1}{10}\frac{x^2}{h} - \frac{5}{36}\frac{x^3}{h^2} + \frac{7}{96}\frac{x^4}{h^3} - \frac{1}{60}\frac{x^5}{h^4} + \frac{1}{720}\frac{x^6}{h^5}\right] f_{n+5} \quad (8)$$

When equation (8) is evaluated at $x = x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ it gives the following discrete block schemes

$$\begin{aligned} y_{n+1} &= y_n + \frac{95}{288}hf_n + \frac{1427}{1440}hf_{n+1} - \frac{133}{240}hf_{n+2} + \frac{241}{720}hf_{n+3} - \frac{173}{1440}hf_{n+4} + \frac{3}{160}hf_{n+5} \\ y_{n+2} &= y_n + \frac{14}{45}hf_n + \frac{43}{30}hf_{n+1} + \frac{7}{45}hf_{n+2} + \frac{7}{45}hf_{n+3} - \frac{1}{15}hf_{n+4} + \frac{1}{90}hf_{n+5} \\ y_{n+3} &= y_n + \frac{51}{160}hf_n + \frac{219}{160}hf_{n+1} + \frac{57}{80}hf_{n+2} + \frac{57}{80}hf_{n+3} - \frac{21}{160}hf_{n+4} + \frac{3}{160}hf_{n+5} \\ y_{n+4} &= y_n + \frac{14}{45}hf_n + \frac{64}{45}hf_{n+1} + \frac{8}{15}hf_{n+2} + \frac{64}{45}hf_{n+3} + \frac{14}{45}hf_{n+4} \\ y_{n+5} &= y_n + \frac{95}{288}hf_n + \frac{125}{96}hf_{n+1} + \frac{125}{144}hf_{n+2} + \frac{125}{144}hf_{n+3} + \frac{125}{96}hf_{n+4} + \frac{95}{288}hf_{n+5} \end{aligned} \quad (9)$$

Derivation of Block Hybrid 5^{th} Step Linear Multistep Method

From equation (2) $m = 8, t = 1$

$$y(x) = \sum_{j=0}^{8} a_j x^j \quad (10)$$

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The collocation equation is

$$y'(x) = \sum_{j=1}^{8} j a_j x^{j-1} \tag{11}$$

Interpolating (10) at $x = x_n$ and collocating (11) at $x = x_{n+j},\ j = 0,1,2,\frac{5}{2},3,\frac{7}{2},4,\frac{9}{2},5$ gives

$$y(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 + a_8 x_n^8 + a_9 x_n^9$$

$$f(x_n) = a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 + 6a_6 x_n^5 + 7a_7 x_n^6 + 8a_8 x_n^7 + 9a_9 x_n^8$$

$$f(x_{n+1}) = a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 + 6a_6 x_{n+1}^5 + 7a_7 x_{n+1}^6 + 8a_8 x_{n+1}^7 + 9a_9 x_{n+1}^8$$

$$f(x_{n+2}) = a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + 4a_4 x_{n+2}^3 + 5a_5 x_{n+2}^4 + 6a_6 x_{n+2}^5 + 7a_7 x_{n+2}^6 + 8a_8 x_{n+2}^7 + 9a_9 x_{n+2}^8$$

$$f\left(x_{n+\frac{5}{2}}\right) = a_1 + 2a_2 x_{n+\frac{5}{2}} + 3a_3 x^2_{n+\frac{5}{2}} + 4a_4 x^3_{n+\frac{5}{2}} + 5a_5 x^4_{n+\frac{5}{2}} + 6a_6 x^5_{n+\frac{5}{2}} + 7a_7 x^6_{n+\frac{5}{2}} + 8a_8 x^7_{n+\frac{5}{2}} + 9a_9 x^8_{n+\frac{5}{2}}$$

$$f(x_{n+3}) = a_1 + 2a_2 x_{n+3} + 3a_3 x_{n+3}^2 + 4a_4 x_{n+3}^3 + a_5 x_{n+3}^4 + 6a_6 x_{n+3}^5 + 7a_7 x_{n+3}^6 + 8a_8 x_{n+3}^7 + 9a_9 x_{n+3}^8$$

$$f\left(x_{n+\frac{7}{2}}\right) = a_1 + 2a_2 x_{n+\frac{7}{2}} + 3a_3 x^2_{n+\frac{7}{2}} + 4a_4 x^3_{n+\frac{7}{2}} + 5a_5 x^4_{n+\frac{7}{2}} + 6a_6 x^5_{n+\frac{7}{2}} + 7a_7 x^6_{n+\frac{7}{2}} + 8a_8 x^7_{n+\frac{7}{2}} + 9a_9 x^8_{n+\frac{7}{2}}$$

$$f(x_{n+4}) = a_1 + 2a_2 x_{n+4} + 3a_3 x_{n+4}^2 + 4a_4 x_{n+4}^3 + 5a_5 x_{n+4}^4 + 6a_6 x_{n+4}^5 + 7a_7 x_{n+4}^6 + + 8a_8 x_{n+4}^7 + 9a_9 x_{n+4}^8$$

$$f\left(x_{n+\frac{9}{2}}\right) = a_1 + 2a_2 x_{n+\frac{9}{2}} + 3a_3 x^2_{n+\frac{9}{2}} + 4a_4 x^3_{n+\frac{9}{2}} + 5a_5 x^4_{n+\frac{9}{2}} + 6a_6 x^5_{n+\frac{9}{2}} + 7a_7 x^6_{n+\frac{9}{2}} + 8a_8 x^7_{n+\frac{9}{2}} + 9a_9 x^8_{n+\frac{9}{2}}$$

$$f(x_{n+5}) = a_1 + 2a_2 x_{n+5} + 3a_3 x_{n+5}^2 + 4a_4 x_{n+5}^3 + 5a_5 x_{n+5}^4 + 6a_6 x_{n+5}^5 + 7a_7 x_{n+5}^6 + + 8a_8 x_{n+5}^7 + 9a_9 x_{n+5}^8$$

(12)

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Arranging (12) in matrix form

$$\begin{pmatrix} 1 & x_n & x_{n^2} & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x^2_{n+\frac{5}{2}} & 4x^3_{n+\frac{5}{2}} & 5x^4_{n+\frac{5}{2}} & 6x^5_{n+\frac{5}{2}} & 7x^6_{n+\frac{5}{2}} & 8x^7_{n+\frac{5}{2}} & 9x^8_{n+\frac{5}{2}} \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 \\ 0 & 1 & 2x_{n+\frac{7}{2}} & 3x^2_{n+\frac{7}{2}} & 4x^3_{n+\frac{7}{2}} & 5x^4_{n+\frac{7}{2}} & 6x^5_{n+\frac{7}{2}} & 7x^6_{n+\frac{7}{2}} & 8x^7_{n+\frac{7}{2}} & 9x^8_{n+\frac{7}{2}} \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 \\ 0 & 1 & 2x_{n+\frac{9}{2}} & 3x^2_{n+\frac{9}{2}} & 4x^3_{n+\frac{9}{2}} & 5x^4_{n+\frac{9}{2}} & 6x^5_{n+\frac{9}{2}} & 7x^6_{n+\frac{9}{2}} & 8x^7_{n+\frac{9}{2}} & 9x^8_{n+\frac{9}{2}} \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & 9x_{n+5}^8 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}$$

$$= \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \\ f_{n+\frac{7}{2}} \\ f_{n+5} \end{pmatrix} \qquad (13)$$

When the unknown parameters $a_{j,}$ $j = 0,1,2,3,4,5,6,7,8,9$ are determined and substituted into (10) yields the continuous formulation as

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$$
\begin{aligned}
y(x) = y_n + \Bigg[& x - \frac{4021}{2520}\frac{x^2}{h} + \frac{159319}{113400}\frac{x^3}{h^2} - \frac{609}{800}\frac{x^4}{h^3} + \frac{2387}{9000}\frac{x^5}{h^4} - \frac{43}{720}\frac{x^6}{h^5} \\
& + \frac{53}{6300}\frac{x^7}{h^6} - \frac{17}{25200}\frac{x^8}{h^7} + \frac{1}{42525}\frac{x^9}{h^8}\Bigg] f_n \\
& + \Bigg[\frac{15}{2}\frac{x^2}{h} - \frac{2761}{1\ 252}\frac{x^3}{h^2} + \frac{10927}{1440}\frac{x^4}{h^3} - \frac{1379}{450}\frac{x^5}{h^4} + \frac{329}{432}\frac{x^6}{h^5} - \frac{29}{252}\frac{x^7}{h^6} \\
& + \frac{7}{720}\frac{x^8}{h^7} - \frac{1}{2835}\frac{x^9}{h^8}\Bigg] f_{n+1} \\
& + \Bigg[-\frac{105}{2}\frac{x^2}{h} + \frac{3391}{36}\frac{x^3}{h^2} - \frac{54227}{720}\frac{x^4}{h^3} + \frac{30437}{900}\frac{x^5}{h^4} - \frac{1969}{216}\frac{x^6}{h^5} + \frac{185}{126}\frac{x^7}{h^6} \\
& - \frac{47}{360}\frac{x^8}{h^7} + \frac{2}{055}\frac{x^9}{h^8}\Bigg] f_{n+2} \\
& + \Bigg[\frac{672}{5}\frac{x^2}{h} - \frac{56272}{225}\frac{x^3}{h^2} + \frac{46846}{225}\frac{x^4}{h^3} - \frac{109208}{1125}\frac{x^5}{h^4} + \frac{3652}{135}\frac{x^6}{h^5} - \frac{7064}{1575}\frac{x^7}{h^6} \\
& + \frac{92}{225}\frac{x^8}{h^7} - \frac{32}{2025}\frac{x^9}{h^8}\Bigg] f_{n+\frac{5}{2}} \\
& + \Bigg[-\frac{175x^2}{h} + \frac{18005}{54}\frac{x^3}{h^2} - \frac{4567}{16}\frac{x^4}{h^3} + \frac{4111}{30}\frac{x^5}{h^4} - \frac{943}{24}\frac{x^6}{h^5} + \frac{281}{42}\frac{x^7}{h^6} \\
& - \frac{5}{8}\frac{x^8}{h^7} + \frac{2}{81}\frac{x^9}{h^8}\Bigg] f_{n+3} \\
& + \Bigg[\frac{960}{7}\frac{x^2}{h} - \frac{16736}{63}\frac{x^3}{h^2} + \frac{10444}{45}\frac{x^4}{h^3} - \frac{25648}{225}\frac{x^5}{h^4} + \frac{904}{27}\frac{x^6}{h^5} - \frac{368}{63}\frac{x^7}{h^6} + \frac{176}{315}\frac{x^8}{h^7} \\
& - \frac{64}{2835}\frac{x^9}{h^8}\Bigg] f_{n+\frac{7}{2}} \\
& + \Bigg[-\frac{525}{8}\frac{x^2}{h} + \frac{9265}{72}\frac{x^3}{h^2} - \frac{32881}{288}\frac{x^4}{h^3} + \frac{4111}{72}\frac{x^5}{h^4} - \frac{7393}{432}\frac{x^6}{h^5} + \frac{769}{252}\frac{x^7}{h^6} \\
& - \frac{43}{144}\frac{x^8}{h^7} + \frac{1}{81}\frac{x^9}{h^8}\Bigg] f_{n+4} \\
& + \Bigg[\frac{160}{9}\frac{x^2}{h} - \frac{19952}{567}\frac{x^3}{h^2} + \frac{158}{5}\frac{x^4}{h^3} - \frac{3608}{225}\frac{x^5}{h^4} + \frac{44}{9}\frac{x^6}{h^5} - \frac{8}{9}\frac{x^7}{h^6} + \frac{4}{45}\frac{x^8}{h^7} \\
& - \frac{32}{8505}\frac{x^9}{h^8}\Bigg] f_{n+\frac{9}{2}} \\
& + \Bigg[-\frac{21}{10}\frac{x^2}{h} + \frac{3769}{900}\frac{x^3}{h^2} - \frac{27341}{7200}\frac{x^4}{h^3} + \frac{2194}{1125}\frac{x^5}{h^4} - \frac{1303}{2160}\frac{x^6}{h^5} + \frac{703}{6300}\frac{x^7}{h^6} \\
& - \frac{41}{3600}\frac{x^8}{h^7} + \frac{1}{2025}\frac{x^9}{h^8}\Bigg] f_{n+5}
\end{aligned}
\tag{14}
$$

When equation (14) is evaluated at $x = x_{n+1}, x_{n+2},\ x_{n+\frac{5}{2}}, x_{n+3}, x_{n+\frac{7}{2}}, x_{n+4,} x_{n+\frac{9}{2}}, x_{n+5},$ it gives the following block schemes

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$$y_{n+1} = y_n + \frac{1777927}{6804000}hf_n + \frac{781673}{453600}hf_{n+1} - \frac{1718021}{226800}hf_{n+2} + \frac{1303586}{70875}hf_{n+\frac{5}{2}} - \frac{1051433}{45360}hf_{n+3} + \frac{251836}{14175}hf_{n+\frac{7}{2}} - \frac{151783}{18144}hf_{n+4} + \frac{95198}{42525}hf_{n+\frac{9}{2}} - \frac{594011}{2268000}hf_{n+5}$$

$$y_{n+2} = y_n + \frac{54847}{212625}hf_n + \frac{57181}{28350}hf_{n+1} - \frac{80077}{14175}hf_{n+2} + \frac{1101472}{70875}hf_{n+\frac{5}{2}} - \frac{57181}{2835}hf_{n+3} + \frac{222272}{14175}hf_{n+\frac{7}{2}} - \frac{4222}{567}hf_{n+4} + \frac{85216}{42525}hf_{n+\frac{9}{2}} - \frac{33367}{141750}hf_{n+5}$$

$$y_{n+\frac{5}{2}} = y_n + \frac{1797625}{6967295}hf_n + \frac{4680625}{2322432}hf_{n+1} - \frac{6339625}{1161216}hf_{n+2} + \frac{579205}{36288}hf_{n+\frac{5}{2}} - \frac{23624375}{1161216}hf_{n+3} + \frac{143125}{9072}hf_{n+\frac{7}{2}} - \frac{17380625}{2322432}hf_{n+4} + \frac{219125}{108864}hf_{n+\frac{9}{2}} - \frac{548915}{2322432}hf_{n+5}$$

$$y_{n+3} = y_n + \frac{21671}{84000}hf_n + \frac{11289}{5600}hf_{n+1} - \frac{15333}{2800}hf_{n+2} + \frac{14178}{875}hf_{n+\frac{5}{2}} - \frac{11209}{560}hf_{n+3} + \frac{2748}{175}hf_{n+\frac{7}{2}} - \frac{1671}{224}hf_{n+4} + \frac{1054}{525}hf_{n+\frac{9}{2}} - \frac{6603}{28000}hf_{n+5}$$

$$y_{n+\frac{7}{2}} = y_n + \frac{32099683}{124416000}hf_n + \frac{16717967}{8294400}hf_{n+1} - \frac{22676759}{4147200}hf_{n+2} + \frac{10471447}{648000}hf_{n+\frac{5}{2}} - \frac{16360757}{829440}hf_{n+3} + \frac{517643}{32400}hf_{n+\frac{7}{2}} - \frac{2486407}{331776}hf_{n+4} + \frac{782971}{388800}hf_{n+\frac{9}{2}} - \frac{9803969}{41472000}hf_{n+5}$$

$$y_{n+4} = y_n + \frac{54854}{212625}hf_n + \frac{28576}{14175}hf_{n+1} - \frac{77624}{14175}hf_{n+2} + \frac{1147904}{70875}hf_{n+\frac{5}{2}} - \frac{56192}{2835}hf_{n+3} + \frac{231424}{14175}hf_{n+\frac{7}{2}} - \frac{4118}{567}hf_{n+4} + \frac{84992}{42525}hf_{n+\frac{9}{2}} - \frac{16672}{70875}hf_{n+5}$$

$$y_{n+\frac{9}{2}} = y_n + \frac{924759}{3584000}hf_n + \frac{1444473}{716800}hf_{n+1} - \frac{1957041}{358400}hf_{n+2} + \frac{903393}{56000}hf_{n+\frac{5}{2}} - \frac{1410723}{71680}hf_{n+3} + \frac{45117}{2800}hf_{n+\frac{7}{2}} - \frac{195777}{28672}hf_{n+4} + \frac{24543}{11200}hf_{n+\frac{9}{2}} - \frac{861111}{3584000}hf_{n+5}$$

$$y_{n+5} = y_n + \frac{2005}{7776}hf_n + \frac{36625}{18144}hf_{n+1} - \frac{50125}{9072}hf_{n+2} + \frac{9290}{567}hf_{n+\frac{5}{2}} - \frac{183125}{9072}hf_{n+3} + \frac{9500}{567}hf_{n+\frac{7}{2}} - \frac{134375}{18144}hf_{n+4} + \frac{4750}{1701}hf_{n+\frac{9}{2}} - \frac{1655}{18144}hf_{n+5}$$

(15)

Reformulation of the Derived Block Schemes into Block Runge-Kutta method

Rearranging $K = 5$ Grid and Hybrid block method in the form $A_0 y_{n+i} = A_0 y_{n-i} + B_0 f_{n+i} + B_0 f_{n-i}$

From equation (9) at Grid K=5

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

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$$\begin{pmatrix} \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} & \frac{3}{160} \\ \frac{43}{30} & \frac{7}{45} & \frac{7}{45} & -\frac{1}{15} & \frac{1}{90} \\ \frac{219}{160} & \frac{57}{80} & \frac{57}{80} & -\frac{21}{160} & \frac{3}{160} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{125}{96} & \frac{125}{144} & \frac{125}{144} & \frac{125}{96} & \frac{95}{288} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{95}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{51}{160} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{95}{288} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{16}$$

Taking the inverse of A_0 gives: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Multiplying A_0, A_1, B_0, B_1 by ${A_0}^{-1}$ to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$
$$\begin{pmatrix} \frac{1427}{1440} & -\frac{133}{240} & \frac{241}{720} & -\frac{173}{1440} & \frac{3}{160} \\ \frac{43}{30} & \frac{7}{45} & \frac{7}{45} & -\frac{1}{15} & \frac{1}{90} \\ \frac{219}{160} & \frac{57}{80} & \frac{57}{80} & -\frac{21}{160} & \frac{3}{160} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} & 0 \\ \frac{125}{96} & \frac{125}{144} & \frac{125}{144} & \frac{125}{96} & \frac{95}{288} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{95}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{51}{160} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & \frac{95}{288} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{17}$$

Equation (17) can be written in Block Runge-Kutta type method as

$$y_{n+1} = y_n + h\left[\frac{95}{288}k_1 + \frac{1427}{1440}k_2 - \frac{133}{240}k_3 + \frac{241}{720}k_4 - \frac{173}{1440}k_5 + \frac{3}{160}k_6\right]$$
$$y_{n+2} = y_n + h\left[\frac{14}{45}k_1 + \frac{43}{30}k_2 + \frac{7}{45}k_3 + \frac{7}{45}k_4 - \frac{1}{15}k_5 + \frac{1}{90}k_6\right]$$
$$y_{n+3} = y_n + h\left[\frac{51}{160}k_1 + \frac{219}{160}k_2 + \frac{57}{80}k_3 + \frac{57}{80}k_4 - \frac{21}{160}k_5 + \frac{3}{160}k_6\right]$$
$$y_{n+4} = y_n + h\left[\frac{14}{45}k_1 + \frac{64}{45}k_2 + \frac{8}{15}k_3 + \frac{64}{45}k_4 + \frac{14}{45}k_5\right]$$
$$y_{n+5} = y_n + h\left[\frac{95}{288}k_1 + \frac{125}{96}k_2 + \frac{125}{144}k_3 + \frac{125}{144}k_4 + \frac{125}{96}k_5 + \frac{95}{288}k_6\right]$$

Where

$k_1 = f_n(x_n, y_n)$

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$$
\begin{aligned}
k_2 &= f(x_n+h), y_n + h\left(\frac{95}{288}k_1 + \frac{1427}{1440}k_2 - \frac{133}{240}k_3 + \frac{241}{720}k_4 - \frac{173}{1440}k_5 + \frac{3}{160}k_6\right)\\
k_3 &= f(x_n+2h), y_n + h\left(\frac{14}{45}k_1 + \frac{43}{30}k_2 + \frac{7}{45}k_3 + \frac{7}{45}k_4 - \frac{1}{15}k_5 + \frac{1}{90}k_6\right)\\
k_4 &= f(x_n+3h), y_n + h\left(\frac{51}{160}k_1 + \frac{219}{160}k_2 + \frac{57}{80}k_3 + \frac{57}{80}k_4 - \frac{21}{160}k_5 + \frac{3}{160}k_6\right)\\
k_5 &= f(x_n+4h), y_n + h\left(\frac{14}{45}k_1 + \frac{64}{45}k_2 + \frac{8}{15}k_3 + \frac{64}{45}k_4 + \frac{14}{45}k_5\right)\\
k_6 &= f(x_n+5h), y_n + h\left(\frac{95}{288}k_1 + \frac{125}{96}k_2 + \frac{125}{144}k_3 + \frac{125}{144}k_4 + \frac{125}{96}k_5 + \frac{95}{288}k_6\right)
\end{aligned}
\tag{18}
$$

From equation (15) at Hybrid $K = 5$

$$
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
y_{n+1}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3}\\ y_{n+\frac{7}{2}}\\ y_{n+4}\\ y_{n+\frac{9}{2}}\\ y_{n+5}
\end{pmatrix}
=
\begin{pmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
y_{n-4}\\ y_{n-3}\\ y_{n-\frac{5}{2}}\\ y_{n-2}\\ y_{n-\frac{3}{2}}\\ y_{n-1}\\ y_{n-\frac{1}{2}}\\ y_n
\end{pmatrix}
+
$$

$$
\begin{pmatrix}
\frac{781673}{453600} & -\frac{1718021}{226800} & \frac{1303586}{70875} & -\frac{1051433}{45360} & \frac{251836}{14175} & -\frac{151783}{18144} & \frac{95198}{42525} & -\frac{594011}{2268000}\\
\frac{57181}{28350} & -\frac{80077}{14175} & \frac{1101472}{70875} & -\frac{57181}{2835} & \frac{222272}{14175} & -\frac{4222}{567} & \frac{85216}{42525} & -\frac{33367}{141750}\\
\frac{4680625}{2322432} & -\frac{6339625}{1161216} & \frac{579205}{36288} & -\frac{23624375}{1161216} & \frac{143125}{9072} & -\frac{17380625}{2322432} & \frac{219125}{108864} & -\frac{548915}{2322432}\\
\frac{11289}{5600} & -\frac{15333}{2800} & \frac{14178}{875} & -\frac{11209}{560} & \frac{2748}{175} & -\frac{1671}{224} & \frac{1054}{525} & -\frac{6603}{28000}\\
\frac{16717967}{8294400} & -\frac{22676759}{4147200} & \frac{10471447}{648000} & -\frac{16360757}{829440} & \frac{517643}{32400} & -\frac{2486407}{331776} & \frac{782971}{388800} & -\frac{9803969}{41472000}\\
\frac{28576}{14175} & -\frac{77624}{14175} & \frac{1147904}{70875} & -\frac{56192}{2835} & \frac{231424}{14175} & -\frac{4118}{567} & \frac{84992}{42525} & -\frac{16672}{70875}\\
\frac{1444473}{716800} & -\frac{1957041}{358400} & \frac{903393}{56000} & -\frac{1410723}{71680} & \frac{45117}{2800} & -\frac{195777}{28672} & \frac{24543}{11200} & -\frac{861111}{3584000}\\
\frac{36625}{18144} & -\frac{50125}{9072} & \frac{9290}{567} & -\frac{183125}{9072} & \frac{9500}{567} & -\frac{134375}{18144} & \frac{4750}{1701} & -\frac{1655}{18144}
\end{pmatrix}
\begin{pmatrix}
f_{n+1}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3}\\ f_{n+\frac{7}{2}}\\ f_{n+4}\\ f_{n+\frac{9}{2}}\\ f_{n+5}
\end{pmatrix}
$$

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$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1777927}{6804000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{54847}{212625} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1797625}{6967295} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{21671}{84000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32099683}{124416000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{54854}{212625} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{924759}{3584000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2005}{7776} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} \tag{19}$$

Taking the inverse the A_0 gives: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Multiplying A_0, A_1, B_0, B_1 by ${A_0}^{-1}$ to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} +$$

$$\begin{pmatrix} \frac{781673}{453600} & -\frac{1718021}{226800} & \frac{1303586}{70875} & -\frac{1051433}{45360} & \frac{251836}{14175} & -\frac{151783}{18144} & \frac{95198}{42525} & -\frac{594011}{2268000} \\ \frac{57181}{28350} & -\frac{80077}{14175} & \frac{1101472}{70875} & -\frac{57181}{2835} & \frac{222272}{14175} & -\frac{4222}{567} & \frac{85216}{42525} & -\frac{33367}{141750} \\ \frac{4680625}{2322432} & -\frac{6339625}{1161216} & \frac{579205}{36288} & -\frac{23624375}{1161216} & \frac{143125}{9072} & -\frac{17380625}{2322432} & \frac{219125}{108864} & -\frac{548915}{2322432} \\ \frac{11289}{5600} & -\frac{15333}{2800} & \frac{14178}{875} & -\frac{11209}{560} & \frac{2748}{175} & -\frac{1671}{224} & \frac{1054}{525} & -\frac{6603}{28000} \\ \frac{16717967}{8294400} & -\frac{22676759}{4147200} & \frac{10471447}{648000} & -\frac{16360757}{829440} & \frac{517643}{32400} & -\frac{2486407}{331776} & \frac{782971}{388800} & -\frac{9803969}{41472000} \\ \frac{28576}{14175} & -\frac{77624}{14175} & \frac{1147904}{70875} & -\frac{56192}{2835} & \frac{231424}{14175} & -\frac{4118}{567} & \frac{84992}{42525} & -\frac{16672}{70875} \\ \frac{1444473}{716800} & -\frac{1957041}{358400} & \frac{903393}{56000} & -\frac{1410723}{71680} & \frac{45117}{2800} & -\frac{195777}{28672} & \frac{24543}{11200} & -\frac{861111}{3584000} \\ \frac{36625}{18144} & -\frac{50125}{9072} & \frac{9290}{567} & -\frac{183125}{9072} & \frac{9500}{567} & -\frac{134375}{18144} & \frac{4750}{1701} & -\frac{1655}{18144} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \\ f_{n+\frac{9}{2}} \\ f_{n+5} \end{pmatrix}$$

$+$

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$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1777927}{6804000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{54847}{212625} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1797625}{6967295} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{21671}{84000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32099683}{124416000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{54854}{212625} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{924759}{3584000} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2005}{7776} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}$$

(20)

Equation (20) can be written Block Runge-Kutta type method as

$$y_{n+1} = y_n + h\left[\frac{1777927}{6804000}k_1 + \frac{781673}{453600}k_2 - \frac{1718021}{226800}k_3 + \frac{1303586}{70875}k_4 - \frac{1051433}{45360}k_5 + \frac{251836}{14175}k_6 - \frac{151783}{18144}k_7 + \frac{95198}{42525}k_8 - \frac{594011}{2268000}k_9\right]$$

$$y_{n+2} = y_n + h\left[\frac{54847}{212625}k_1 + \frac{57181}{28350}k_2 - \frac{80077}{14175}k_3 + \frac{1101472}{70875}k_4 - \frac{57181}{2835}k_5 + \frac{222272}{14175}k_6 - \frac{4222}{567}k_7 + \frac{85216}{42525}k_8 - \frac{33367}{141750}k_9\right]$$

$$y_{n+\frac{5}{2}} = y_n + h\left[\frac{1797625}{6967295}k_1 + \frac{4680625}{2322432}k_2 - \frac{6339625}{1161216}k_3 + \frac{579205}{36288}k_4 - \frac{23624375}{1161216}k_5 + \frac{143125}{9072}k_6 - \frac{17380625}{2322432}k_7 + \frac{219125}{108864}k_8 - \frac{548915}{2322432}k_9\right]$$

$$y_{n+3} = y_n + h\left[\frac{21671}{84000}k_1 + \frac{11289}{5600}k_2 - \frac{15333}{2800}k_3 + \frac{14178}{875}k_4 - \frac{11209}{560}k_5 + \frac{2748}{175}k_6 - \frac{1671}{224}k_7 + \frac{1054}{525}k_8 - \frac{6603}{28000}k_9\right]$$

$$y_{n+\frac{7}{2}} = y_n + h\left[\frac{32099683}{124416000}k_1 + \frac{16717967}{8294400}k_2 - \frac{22676759}{4147200}k_3 + \frac{10471447}{648000}k_4 - \frac{16360757}{829440}k_5 + \frac{517643}{32400}k_6 - \frac{2486407}{331776}k_7 + \frac{782971}{388800}k_8 - \frac{9803969}{41472000}k_9\right]$$

$$y_{n+4} = y_n + h\left[\frac{54854}{212625}k_1 + \frac{28576}{14175}k_2 - \frac{77624}{14175}k_3 + \frac{1147904}{70875}k_4 - \frac{56192}{2835}k_5 + \frac{231424}{14175}k_6 + -\frac{4118}{567}k_7 + \frac{84992}{42525}k_8 - \frac{16672}{70875}k_9\right]$$

$$y_{n+\frac{9}{2}} = y_n + h\left[\frac{924759}{3584000}k_1 + \frac{1444473}{716800}k_2 - \frac{1957041}{358400}k_3 + \frac{903393}{56000}k_4 - \frac{1410723}{71680}k_5 + \frac{45117}{2800}k_6 - \frac{195777}{28672}k_7 + \frac{24543}{11200}k_8 - \frac{861111}{3584000}k_9\right]$$

$$y_{n+5} = y_n + h\left[\frac{2005}{7776}k_1 + \frac{36625}{18144}k_2 - \frac{50125}{9072}k_3 + \frac{9290}{567}k_4 - \frac{183125}{9072}k_5 + \frac{9500}{567}k_6 - \frac{134375}{18144}k_7 + \frac{4750}{1701}k_8 - \frac{1655}{18144}k_9\right]$$

Where

$k_1 = f(x_n, y_n)$

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$k_2 = f(x_n + h), y_n + h\left[\frac{1777927}{6804000}k_1 + \frac{781673}{453600}k_2 - \frac{1718021}{226800}k_3 + \frac{1303586}{70875}k_4 - \frac{1051433}{45360}k_5 + \frac{251836}{14175}k_6 - \frac{151783}{18144}k_7 + \frac{95198}{42525}k_8 - \frac{594011}{2268000}k_9\right]$

$k_3 = f(x_n + 2h), y_n + h\left[\frac{54847}{212625}k_1 + \frac{57181}{28350}k_2 - \frac{80077}{14175}k_3 + \frac{1101472}{70875}k_4 - \frac{57181}{2835}k_5 + \frac{222272}{14175}k_6 - \frac{4222}{567}k_7 + \frac{85216}{42525}k_8 - \frac{33367}{141750}k_9\right]$

$k_4 = f\left(x_n + \frac{5}{2}h\right), y_n + h\left[\frac{1797625}{6967295}k_1 + \frac{4680625}{2322432}k_2 - \frac{6339625}{1161216}k_3 + \frac{579205}{36288}k_4 - \frac{23624375}{1161216}k_5 + \frac{143125}{9072}k_6 - \frac{17380625}{2322432}k_7 + \frac{219125}{108864}k_8 - \frac{548915}{2322432}k_9\right]$

$k_5 = f(x_n + 3h), y_n + h\left[\left[\frac{21671}{84000}k_1 + \frac{11289}{5600}k_2 - \frac{15333}{2800}k_3 + \frac{14178}{875}k_4 - \frac{11209}{560}k_5 + \frac{2748}{175}k_6 - \frac{1671}{224}k_7 + \frac{1054}{525}k_8 - \frac{6603}{28000}k_9\right]\right]$

$k_6 = f\left(x_n + \frac{7}{2}h\right), y_n + h\left[\frac{32099683}{124416000}k_1 + \frac{16717967}{8294400}k_2 - \frac{22676759}{4147200}k_3 + \frac{10471447}{648000}k_4 - \frac{16360757}{829440}k_5 + \frac{517643}{32400}k_6 - \frac{2486407}{331776}k_7 + \frac{782971}{388800}k_8 - \frac{9803969}{41472000}k_9\right]$

$k_7 = f(x_n + 4h), y_n + h\left[\frac{54854}{212625}k_1 + \frac{28576}{14175}k_2 - \frac{77624}{14175}k_3 + \frac{1147904}{70875}k_4 - \frac{56192}{2835}k_5 + \frac{231424}{14175}k_6 - \frac{4118}{567}k_7 + \frac{84992}{42525}k_8 - \frac{16672}{70875}k_9\right]$

$k_8 = f\left(x_n + \frac{9}{2}h\right), y_n + h\left[\frac{924759}{3584000}k_1 + \frac{1444473}{716800}k_2 - \frac{1957041}{358400}k_3 + \frac{903393}{56000}k_4 - \frac{1410723}{71680}k_5 + \frac{45117}{2800}k_6 - \frac{195777}{28672}k_7 + \frac{24543}{11200}k_8 - \frac{861111}{3584000}k_9\right]$

$k_9 = f(x_n + 5h), y_n + h\left[\frac{2005}{7776}k_1 + \frac{36625}{18144}k_2 - \frac{50125}{9072}k_3 + \frac{9290}{567}k_4 - \frac{183125}{9072}k_5 + \frac{9500}{567}k_6 - \frac{134375}{18144}k_7 + \frac{4750}{1701}k_8 - \frac{1655}{18144}k_9\right]$

(21)

Numerical problems

Two distinct problems were used to evaluate the efficiency of the method for Grid and Hybrid Block R-K schemes.

Problem 1:

$xy' + 2y = 4x^2$

$y(1) = 2$

Exact solution $y(x) = \frac{x^2+1}{x^2}$

Problem 2:

$y' = 1 - y^2$

$y(0) = 0, h = 0.1$

Exact solution $y(x) = \tanh(x)$

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Table 1: Numerical Solutions and Approximate Errors of Problem 1 at $K = 5$

x	Exact solution	$K = 5$ Grid Block R-K Method	$K = 5$ Hybrid Block R-K Method	Errors at $K = 5$ Grid Block R-K Method	Errors at $K = 5$ Hybrid Block R-K Method
1.1	2.036446280991740	2.036437026986660	2.036446097592800	9.25400508×10^{-6}	1.8339894×10^{-7}
1.2	2.134444444444440	2.134439354710220	2.134444303705050	5.08973422×10^{-6}	1.4073939×10^{-7}
1.3	2.281715976331360	2.281709880999800	2.281715856230660	6.09533156×10^{-6}	1.201007×10^{-7}
1.4	2.470204081632650	2.470201155148380	2.470203978134010	2.92648427×10^{-6}	1.0349864×10^{-7}
1.5	2.694444444444440	2.694434935769770	2.694444355991990	9.50867467×10^{-6}	8.845245×10^{-7}
1.6	2.950625000000000	2.950616226870890	2.950624918991070	8.77312911×10^{-6}	8.100893×10^{-8}
1.7	3.236020761245670	3.236013113404860	3.236020689730690	7.64784081×10^{-6}	7.151498×10^{-8}
1.8	3.548641975308640	3.548635071104030	3.548641911515640	6.90420461×10^{-6}	6.3793×10^{-8}
1.9	3.887008310249310	3.887002224964390	3.887008252995690	6.08528492×10^{-6}	5.725362×10^{-8}
2.0	4.250000000000000	4.249994170187140	4.249999948361170	5.82981286×10^{-6}	5.163883×10^{-8}
2.1	4.636757369614510	4.636752039898710	4.636757322614230	5.3297158×10^{-6}	4.700028×10^{-8}
2.2	5.046611570247930	5.046606726641960	5.046611527435600	4.84360597×10^{-6}	4.281233×10^{-8}
2.3	5.479035916824200	5.479031476727760	5.479035877653560	4.44009644×10^{-6}	3.917064×10^{-8}
2.4	5.933611111111110	5.933607044922730	5.933611075136720	4.06618838×10^{-6}	3.597439×10^{-8}
2.5	6.410000000000000	6.409996217021300	6.409999966847650	3.7829787×10^{-6}	3.315235×10^{-8}

Table 2: Numerical Solutions and Approximate Errors of Problem 2 at $K = 5$

x	Exact solution	$K = 5$ Grid Block R-K Method	$K = 5$ Hybrid Block R-K Method	Errors at $K = 5$ Grid Block R-K Method	Errors at $K = 5$ Hybrid Block R-K Method
0.1	0.099667994624956	0.099667842156925	0.099667997591513	1.52468031×10^{-7}	2.966557×10^{-9}
0.2	0.197375320224904	0.197375212238800	0.197375322877983	1.07986104×10^{-7}	2.653079×10^{-9}
0.3	0.291312612451591	0.291312486795530	0.291312614981234	1.25656061×10^{-7}	2.529643×10^{-9}
0.4	0.379948962255225	0.379948869788473	0.379948964619416	9.2466752×10^{-8}	2.364191×10^{-9}
0.5	0.462117157260010	0.462117015227449	0.462117159402163	1.42032561×10^{-7}	2.142153×10^{-9}
0.6	0.537049566998036	0.537049593136490	0.537049567901442	2.6138454×10^{-8}	9.03406×10^{-10}
0.7	0.604367777117163	0.604367753533472	0.604367778001880	2.3583691×10^{-8}	8.84717×10^{-10}
0.8	0.664036770267849	0.664036781368525	0.664036771045971	1.1100676×10^{-8}	7.78122×10^{-10}
0.9	0.716297870199025	0.716297836292514	0.716297870877112	3.3906511×10^{-8}	6.78087×10^{-10}
1.0	0.761594155955765	0.761594260500233	0.761594156551862	1.04544468×10^{-7}	5.96097×10^{-10}
1.1	0.800499021760631	0.800499085275385	0.800499022461743	6.3514754×10^{-8}	7.01112×10^{-10}
1.2	0.833654607012155	0.833654669699617	0.833654607592362	6.2687462×10^{-8}	5.80207×10^{-10}
1.3	0.861723159313306	0.861723205675963	0.861723159803198	4.6362657×10^{-8}	4.89892×10^{-10}
1.4	0.885351648202262	0.885351696888310	0.885351648613516	4.8686048×10^{-8}	4.11254×10^{-10}

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1.5	0.905148253644867	0.905148261876376	0.905148253986190	8.231509×10^{-8}	3.41323×10^{-10}

Discussion

This study uses two test problems: a linear and non-linear to validate the proposed methods. A comparison of the results presented in Tables 1 and 2, shows that the hybrid block R-K methods converges faster than grid block R-K methods for both cases.

Conclusion

Numerical experiments showed that the hybrid block R-K method achieves faster converges than the grid block R-K method. Both methods showed computational speed. Overall, this study concludes that applying strategies to block R-K methods can greatly boost the efficiency and accuracy of linear and nonlinear first-order ODEs.

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