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Lie Symmetry Analysis, Exact Solutions, and Conservation Laws of the Geophysical Korteweg–de Vries Equation

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Abstract
The Korteweg-de Vries equation is a nonlinear PDE that is used to describe most physical systems involving dispersion, such as wave propagation, fluid dynamics, and plasma physics. In the light of the influence of coriolis effect on waves, the study of the Geophysical Korteweg-de Vries (GKdV) equation is examined. The Lie point symmetries and conservation laws of the equation are constructed and with the Lie point symmetries, a symmetry analysis is performed to reduce the equation to an integrable form. Numerical solutions of the reduced equation were considered for the travelling wave (periodic wave) of the GKdV equation for the parameters δ (for the coriolis effect) and λ (for the velocity of the wave). To examine the coriolis effect on free flow in oceans, the dynamical system analysis is applied on the GKdV equation. From the study, it is revealed that travelling wave velocity and coriolis factor have significant effects on the transmission of the periodic wave solution of the GKdV equation. The results obtained stands as a motivation to extend the method to some other nonlinear evolution equations.

Keywords: Korteweg-de Vries Equation, Lie Symmetry Analysis, Conservation Laws, Exact solution, Dynamical Systems.

Introduction

"Nonlinear evolution equations (NLEEs), that is, dynamical partial differential equations, that deal with both time and space as independent variables, have contributed immensely within the disciplines of Mathematics and Physics, where this is greatly reflected in nonlinear physical systems (Olver, 2014;Rizvi et al., 2020)." The nonlinear physical systems include fluid dynamics, plasma physics, wave propagation and many more. There is a notable NLEE that is recognized for the majority of scientific, biological, and chemical issues, such as quantum field theory and solid mechanics (Rizvi et al., 2021), among others which is known as Nonlinear Partial Differential Equation (NLPDE) (Ablowitz &Clarkson, 1991).

As one of the partial differential equations that are nonlinear, the Korteweg-deVries equation

$$\rho_t + \alpha\rho\rho_x + \beta\rho_{xxx} = 0 \qquad 1.1$$

generally is regarded as the equation that describes how waves with tiny amplitudes propagate unidirectionally in a nonlinear dispersive medium (Miura, 1976), and is well known as a typical example to describe long waves that are weakly nonlinear in numerous engineering and scientific fields (Xiang, 2015).

The KdV equation was first developed as a comprehensive analysis of canal waves in shallow water (Xiang, 2015). "However, the discovery of shallow waves is credited to John Scott Russell, who in 1834 made the first observation of the isolated wave, a long water wave with no change in shape that he named the enormous translation wave that crossed the Canal of Edinburgh–Glasgow (Debnath, 2012; Shingareva & Lizarraga-Celaya, 2011)." However, it was Korteweg and de Vries who in 1895 "looked into the deformation of a system of waves of arbitrary shape but moving in one direction only, wherein they obtained the differential equation for stationary waves (Korteweg & de Vries, 1895)."

As advancements were made in the field of nonlinear sciences, (Rizvi et al., 2020), several methods were introduced in other to determine “the exact solution of Partial Differential Equations (PDEs) that are nonlinear such as the tanh-function method, the extended tanh-function method, the sine-cosine method, the (G’/G)-expansion method and the lie symmetry analysis (Jafari et al., 2013).” “Lie symmetry and conservation laws are important instruments for deciphering how a physical system behaves and for solving diverse problems pertaining to mathematical physics (Okeke et al., 2019).”

According to (Baleanu et al., 2017), Lie symmetry analysis, introduced by Sophus Lie a Norwegian Mathematician (1842 – 1899) (Krishnakunar et al., 2020) as one of the efficient techniques used in investigating solutions of nonlinear partial differential equations that are exact, “applies continuous transformation groups to determine invariant and exact solutions of differential equations, that enables one to obtain differential equation solutions that are entirely algorithmic (Bluman et al., 2010; Olivieri, 2010).” “The Lie groups of point transformations, which are defined by infinitesimal generators, are fundamental to the study of Lie symmetry analysis (Bluman et al., 2010).” According to (Bluman et al., 2010), the process of solving associated linear systems of equations to be determined for the infinitesimal generators reduces the challenge of identifying the Lie group of point transformations that leaves invariant a differential equation (partial or ordinary). Furthermore, they posited that “Sophus Lie demonstrated that, point symmetry of an Ordinary Differential Equation (ODE) causes the ODE's order to reduce or a reduction in the number of independent variables for a Partial Differential Equation (PDE) (Bluman et al., 2010).”

For their part, (Shingareva & Lizarraga-Celaya, 2011) stated that Lie group also referred to as continuous group is an approach that is centered on finding the symmetries of differential equations where an explicit computational approach used to calculate the continuous group of point transformations for a given differential equation, whether it is linear or nonlinear is known as Lie group analysis. This Lie group analysis (Lie symmetry analysis) is an algebraic approach that is based on transformation methods which allows us to identify transformations wherefore we have invariance of a nonlinear PDE, and introduce new independent and dependent variables that simplify the differential equation. Sophus Lie as stated in the work of (Shingareva & Lizarraga-Celaya, 2011), outlined that the process for determining symmetries of differential equation, that is, determining transformations that preserve the equations' structure, is comprised of four stages: firstly, to present the set of transformations with a single parameter, secondly, give the invariance condition which will have a polynomial representation, thirdly, obtaining the determining system which is solved and then find the coordinates of the infinitesimal operator, and fourthly, to consider “if this determining equations can acknowledge other solutions that will produce other infinitesimal operators (Shingareva & Lizarraga-Celaya, 2011).”

“Conservation laws are divergence expressions that disappear at partial differential equation solutions. They are essential in relation to the decrease and solution of partial differential equations; in particular, a strong integrability of the partial differential equation indicates several conservation laws pertaining to partial differential equation (Naz, 2012; Bluman & Anco, 2002).” “The major step in determining the exact solution of nonlinear PDEs is the derivation of conservation laws, which uncover underlying invariant features (Majola et al., 2021).” Conservation laws have various significant uses, such as investigating the integrability and linearization maps, as well as proving that solutions exist and are unique (Bluman et al., 2010). “There are various approaches available for the computation of conservation laws of DEs. These include, the direct construction method (multiplier approach, variational derivatives approach), symmetry/adjoint symmetry pair method, symmetry action on a known conservation law method, Cheviakov’s recursion formular, and Ibragimov’s conservation theorem (Buhe et al., 2018).”

The direct construction method is an algorithmic approach presented by (Bluman & Anco, 2002) to obtain, irrespective of how many dependent and independent variables there is, partial differential equations’ conservation laws. By using this procedure, it is not necessary to employ or have a variational principle. However, the method demonstrates how to proceed directly with establishing partial differential equations’ conservation laws, having variational principle by utilizing the PDE's symmetries. Given that a PDE's symmetries fulfill a set of linear determining equations and that the PDE and its symmetries are subject to an invariance condition. Then, with an algorithmic calculation, it is possible to verify the invariance condition and which also yields a conservation law construction formula where it is directly derived from the PDE and symmetry.

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However, when it comes to a PDE not having a "variational principle, the approach consists of substituting symmetries with PDE adjoint symmetries that satisfy linear determining equations, that is, the adjoint of the determining equation for symmetries (Bluman & Anco, 2002)." There is an equivalent direct method for getting the conservation laws with regards to the PDE's adjoint symmetries, since the adjoint invariance condition on adjoint symmetries replaces the invariance condition on symmetries. Consequently, a general direct computational technique for establishing the laws for local conservation for specified PDEs is provided by the system of conservation law determination and the conservation law construction formula.

The direct method is a methodical process wherein native conservation laws are constructed whereby with respect to a PDE system $\emptyset\{x;\rho\}$ of order k, we look for multiplier sets (factors, characteristics) of the type:

$$\{\Lambda_\alpha(x,\mathcal{U},\partial\mathcal{U},\dots,\partial^p\mathcal{U}\}_{\alpha=1}^{N}$$

according to a predetermined order p. Next, we ensure the multipliers' dependence on their arguments to forestall the occurrence of singular multipliers. To locate all of these sets of multipliers, a set of determining equations is therefore solved. Next, we determine the associated fluxes.

$$\psi^i(x,\mathcal{U},\partial\mathcal{U},\dots,\partial^r\mathcal{U})$$

that satisfies the identity

$$\Lambda_\alpha(x,\mathcal{U},\partial\mathcal{U},\dots,\partial^p\mathcal{U})\mathcal{R}^\alpha(x,\mathcal{U},\partial\mathcal{U},\dots,\partial^p\mathcal{U}) \equiv \mathfrak{D}_i\omega^i(x,\mathcal{U},\partial\mathcal{U},\dots,\partial^r\mathcal{U}).$$

Finally, a local conservation law is produced by respective set of fluxes and multipliers.

$$\mathfrak{D}_i\omega^i(x,\rho,\partial\rho,\dots,\partial^r\rho) = 0.$$

According to (Liu et al., 2012), among the partial differential equations that are nonlinear that is studied the most, is the KdV equation, which possess remarkable feature: travelling wave solutions, known as solitons. Through its numerical study by (Zabusky & Kruskal, 1965), the KdV equation was found to possess multi-soliton solutions in which the individual solitons move apart without changing their forms after interacting nonlinearly up close. There are several practical applications of the KdV equation which includes but not limited to waves in "bubbly fluids, internal oceanic and atmospheric waves, ion-acoustic waves in collisionless plasma, and shallow-water gravity waves (El, 2007)."

There are several modifications of the Korteweg-de Vries equation. One such modification according to "(Ak et al., 2020) is the geophysical Korteweg-de Vries equation", given as

$$\rho_t - \delta\rho_x + \frac{3}{2}\rho\rho_x + \frac{1}{6}\rho_{xxx} = 0 \qquad 1.2$$

which is primarily used in the study of coriolis effect in relation to oceanic flows, where (Rizvi et al., 2021) posited that "u indicates the advancement of the surface that is free, and δ stands for the Coriolis effect factor." With regard to (1.2), (Karunaker & Chakravety, 2019) employed the Homotopy Pertubation Method (HPM) in working out the nonlinear geophysical Korteweg – de Vries equation's solution. On their part, (Rizvi et al., 2020), investigated the implementation of the Unified Method on (1.2) with respect to extracting the equation's solutions in terms of rational and polynomial functions which degenerates to providing wave solutions such as solitary, soliton, and elliptic wave solutions.. With the aid of (1.2), (Ak et al., 2020), examined the impact of Coriolis effect on oceanic flows where they noted that velocity of travelling waves and Coriolis parameter have major impact on the propagation of single-wave solution.

Statement of problem

The propagation of unidirectional waves in shallow water is described by the Korteweg-de Vries equation that admits an exact solution known as the soliton. There is no specific method for figuring out nonlinear partial differential equations' exact solutions. But determining the equations' exact solutions is a crucial task in nonlinear science. To this end, an amalgamation of the lie symmetry analysis with dynamical system technique is applied in determining the exact solution of the geophysical Korteweg-de Vries equation (1.2).

Methods and Materials

"The Lie Symmetry analysis of differential equation is a (Okeke et al., 2018)" technique that is based on the "Lie groups of point transformations, and which are characterized by infinitesimal generators (Bluman & Anco, 2002)." With this technique, the Lie symmetries of (1.2) are determined, while the conservation laws for the equation are determined by the use of the direct method technique. "The direct construction method as an algorithmic approach is a technique to obtain, irrespective of how many dependent and independent variables there is, conservation laws of partial differential equations (Bluman et al., 2010)."

Lie point symmetry of equation

The Lie point symmetries of the geophysical Korteweg – de Vries equation (1.2) are derived from the vector field that takes the shape

$$X = \tau(t,x,\rho)\frac{\partial}{\partial t} + \zeta(t,x,\rho)\frac{\partial}{\partial x} + \eta(t,x,\rho)\frac{\partial}{\partial \rho}. \quad (2.1)$$

in which the functions of coefficient, $\tau(t,x,\rho)$, $\zeta(t,x,\rho)$, $\eta(t,x,\rho)$, are to be established.
The operator, X, meets the criteria for Lie symmetry given as

$$X^{[3]}[\rho_t - \delta\rho_x + \tfrac{3}{2}\rho\rho + \tfrac{1}{6}\rho_{xxx}]|_{(1.2)} = 0 \quad (2.2)$$

where $X^{[3]}$ denotes the third extension of the operator X and which is defined as

$$X^{[3]} = X + \varsigma_t\frac{\partial}{\partial \rho_t} + \varsigma_x\frac{\partial}{\partial \rho_x} + \varsigma_{xx}\frac{\partial}{\partial \rho_{xx}} + \varsigma_{xxx}\frac{\partial}{\partial \rho_{xxx}} \quad (2.3)$$

and the coefficients ς_t, ς_x, $\varsigma_{xx}$and $\varsigma_{xxx}$are given by

$$\varsigma_t = \mathfrak{D}_t(\eta) - \rho_t\mathfrak{D}_t(\tau) - \rho_x\mathfrak{D}_t(\zeta)$$
$$\varsigma_x = \mathfrak{D}_x(\eta) - \rho_t\mathfrak{D}_x(\tau) - \rho_x\mathfrak{D}_x(\zeta)$$
$$\varsigma_{xx} = \mathfrak{D}_x(\varsigma_x) - \rho_{tx}\mathfrak{D}_x(\tau) - \rho_{xx}\mathfrak{D}_x(\zeta)$$
$$\varsigma_{xxx} = \mathfrak{D}_x(\varsigma_{xx}) - \rho_{txx}\mathfrak{D}_x(\tau) - \rho_{xxx}\mathfrak{D}_x(\zeta)$$

Here $\mathfrak{D}_t, \mathfrak{D}_x$ signify the sum of the derivative operators as specified by

$$\mathfrak{D}_t = \frac{\partial}{\partial t} + \rho_t\frac{\partial}{\partial \rho} + \rho_{xt}\frac{\partial}{\partial \rho_x} + \cdots, \mathfrak{D}_x = \frac{\partial}{\partial x} + \rho_x\frac{\partial}{\partial \rho} + \rho_{tx}\frac{\partial}{\partial \rho_t} + \cdots, \quad (2.4)$$

When (2.2) is expanded and divided according to various derivatives of powers of u, a system that is over determined in the unknown coefficients $\tau(t,x,\rho), \zeta(t,x,\rho)$ and $\eta(t,x,\rho)$ is produced. However, because of its extensive computations, the over determined system cannot be displayed here. Solving the over determined system for $\tau(t,x,\rho), \zeta(t,x,\rho)$ and $\eta(t,x,\rho)$, we obtain

$$\tau(t,x,\rho) = c_1 t + c_2,$$
$$\zeta(t,x,\rho) = \tfrac{1}{3}c_1 x + c_3 t + c_4 \quad (2.5)$$
$$\eta(t,x,\rho) = \tfrac{2}{9}(2\delta - 3\rho)c_1 + \tfrac{2}{3}c_3$$

where c_1, c_2, c_3, c_4denote arbitrary constants. From the equations of (2.5), we were able to get a four-dimensional Lie algebra which was spanned by the subsequent basis

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = t\frac{\partial}{\partial x} + \frac{2}{3}\frac{\partial}{\partial \rho}, X_4 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2}{9}(2\delta - 3\rho)\frac{\partial}{\partial \rho}.$$

Conservation laws of the gkdv equation

A conserved vector that matches a conservation law of equation (1.2), is a 2 - tuple (T^t, T^x), so that

$$\mathfrak{D}_t T^t + \mathfrak{D}_x T^x = 0$$

along the equation's solutions.
The equation (1.2) belongs to the third order partial differential equation, and its conservation laws cannot be derived straight out of a variational principle. We utilize the multiplier approach to examine the conservation laws. A

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multiplier Λ of order up to three, viz., $\Lambda = (\mathrm{t}, x, \rho_x, \rho_t, \rho_{xx}, \rho_{xt}, \rho_{tt}, \rho_{xxx}, \rho_{txx}, \rho_{ttx})$, is taken into consideration for (1.2). The divergence condition

$$\mathfrak{D}_t T^t + \mathfrak{D}_x T^x = \Lambda\left(\rho_t - \delta\rho_x + \frac{3}{2}\rho\rho_x + \frac{1}{6}\rho_{xxx}\right) = 0,$$

is satisfied by the conserved vector (T^t, T^x), of (1.2). (Okeke et al., 2019)

We consider the multiplier, Λ of order zero to two in derivatives with respect to u. We present the multipliers Λ together with the corresponding preserved vector (T^t, T^x) below. In case we look for more higher-order multipliers, the number can be limitless.

$$\Lambda_1 = 1$$
$$T^t = \rho$$
$$T^x = \frac{3}{4}\rho^2 - \delta\rho + \frac{1}{6}\rho_{xx}$$
$$\Lambda_2 = \rho$$
$$T^t = \frac{1}{2}\rho^2$$
$$T^x = \frac{1}{2}\rho^3 - \delta\rho^2 - \frac{1}{12}{\rho_x}^2 + \frac{1}{6}\rho\rho_{xx}$$
$$\Lambda_3 = \frac{9}{2}\rho^2 + \rho_{xx}$$
$$T^t = \frac{3}{2}\rho^3 + \frac{1}{2}\rho\rho_{xx}$$
$$T^x = -\frac{3}{2}\rho^3 - \frac{27}{32}\rho^5\rho_{xx} + \frac{3}{4}\rho^2\rho_{xx} - \frac{3}{2}\delta\rho^3 - \frac{1}{2}\rho\rho_{xt} + \frac{1}{12}{\rho_{xx}}^2 + \frac{1}{2}\rho_x\rho_t - \frac{1}{2}\delta{\rho_x}^2$$
$$\Lambda_4 = 2\delta t - 3t\rho + 2x$$
$$T^t = -\frac{3}{2}t\rho^2 + 2\delta t\rho + 2x\rho$$
$$T^x = -\frac{3}{2}t\rho^3 - \frac{1}{2}t\rho\rho_{xx} + 3\delta t\rho^2 + \frac{1}{4}t{\rho_x}^2 + \frac{3}{2}x\rho^2 + \frac{1}{3}\delta t\rho_{xx} - 2\delta^2 t\rho + \frac{1}{3}x\rho_{xx} - \frac{1}{3}\rho_x - 2\delta x\rho$$

The conserved densities, T^t and fluxes , T^x for the multipliers Λ_5 and Λ_6 below are too cumbersome to present here.

$$\Lambda_5 = -\frac{15}{4}\rho^3 - \rho\rho_{xx} + \frac{1}{4}{\rho_x}^2 + \rho_{xt} + \frac{9}{2}\delta\rho^2$$
$$\Lambda_6 = 15\delta\rho^3 - \frac{105}{16}\rho^4 + \rho\rho_{xt} + \frac{1}{4}\rho{\rho_x}^2 - \frac{1}{4}{\rho_{xx}}^2 - \frac{11}{4}\rho^2\rho_{xx} + 3\delta\rho\rho_{xx} - \rho_t\rho_x - 2\rho_{tt} - 9\delta^2\rho^2.$$

Symmetry reductions and exact solutions of the equation (1.2)

In the present section, we employ the Lie point symmetries obtained in section 2.1 to transform the variables of the equation (1.2) into new similarity variables. With the new similarity variables, the equation (1.2) is reduced to ordinary differential equations for the purpose of determining their exact solutions where possible.

(i) Invariance under $X_1 = \frac{\partial}{\partial t}$

Solving the characteristic equation

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\rho}{0} \tag{3.1}$$

yields

$$z = x, \;\; w(z) = \rho \tag{3.2}$$

for similarity variables of X_1.

The third order ODE

$$-\delta w' + \frac{3}{2}\, w\, w' + \frac{1}{6}w''' = 0 \tag{3.3}$$

is the result of reducing the GKDV equation (1.2) by these similarity variables.
Integrating (3.3) above we obtain

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$$-\delta \int \frac{dw}{dz} dz + \frac{3}{2} w \int \frac{dw}{dz} dz + \frac{1}{6} \frac{d}{dz} \left\{ \frac{d^2 w}{dz^2} \right\} = 0$$
$$-\delta \int dw + \frac{3}{2} w \int dw + \frac{1}{6} \int \frac{d}{dz} \left\{ \frac{d^2 w}{dz^2} \right\} dz = 0$$
$$-\delta \int dw + \frac{3}{2} w \int dw + \frac{1}{6} \int d \left\{ \frac{d^2 w}{dz^2} \right\} = 0$$
$$-\delta w + \frac{3}{2} \int w \, dw + \frac{1}{6} \int d \left\{ \frac{d^2 w}{dz^2} \right\} = 0$$
$$-\delta w + \frac{3}{2} \cdot \frac{w^2}{2} + \frac{1}{6} w'' = c_1$$
$$-\delta w + \frac{3}{4} w^2 + \frac{1}{6} w'' = c_1 \qquad (3.4)$$

Where c_1 is a constant of integration.

(ii) Invariance under $X_2 = \frac{\partial}{\partial x}$

Solving the characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{d\rho}{0}, \qquad (3.5)$$

yields

$$z = t, w(z) = \rho \qquad (3.6)$$

for similarity variables of X_2.

The first order ODE

$$w' = 0 \qquad (3.7)$$

is the result of reducing the equation (1.2) by these similarity variables.

Equation (3.7) implies that

$$w(z) = c_2 \qquad (3.8)$$

where c_2 is a constant. Equation (3.8) is a trivial solution and is not of interest.

(iii) Invariance under $X_3 = t \frac{\partial}{\partial x} + \frac{2}{3} \frac{\partial}{\partial \rho}$

In solving the characteristic equation

$$\frac{dt}{0} = \frac{dx}{c} = \frac{3d\rho}{2} \qquad (3.9)$$

the similarity variables for X_2 are obtained as

$$z = t, w(z) = \frac{2x}{3t} - \rho \qquad (3.10)$$

The first order ODE

$$zw' + w - \frac{2}{3}\delta = 0 \qquad (3.11)$$

is the result of reducing the equation (1.2) by these similarity variables.

The solution to the above equation (3.11) is

$$w(z) = \frac{2}{3}\delta + \frac{c}{t} = 0 \qquad (3.12)$$

where c might be any constant.

In the original variables t, x solution (3.12) becomes

$$\rho(t,x) = \frac{2}{3}\delta + \frac{c}{t} + \frac{2x}{3t} = 0 \qquad (3.13)$$

(iv) Invariance under $X_4 = \frac{1}{3} x \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{2}{9} (2\delta - 3\rho) \frac{\partial}{\partial \rho}$

The characteristic equation for X_4 is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\rho}{0} \qquad (3.14)$$

Solving the above equation (3.14) gives rise to the similarity variables

$$z = \frac{x}{t^{\frac{1}{3}}}, w(z) = \frac{1}{3} t^{\frac{2}{3}} (3\rho - 2\delta) \qquad (3.15)$$

The third order ODE

$$-2zw' + w(-4 + 9w') + w''' = 0 \qquad (3.16)$$

is the result of reducing the equation (1.2) by these similarity variables.

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It is important to acknowledge that the reduced equations (3.4) and (3.16) provide significant challenges in their analytical solution due to their considerable nonlinearity. Numerical solutions of the reduced equations would be considered as the next logical step.

Figure 1

Graph of the Solution 3.13 with $\delta = \lambda = 1$

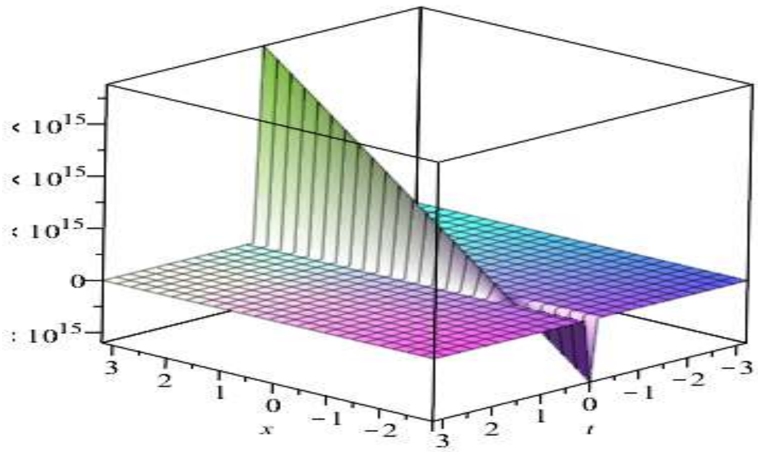
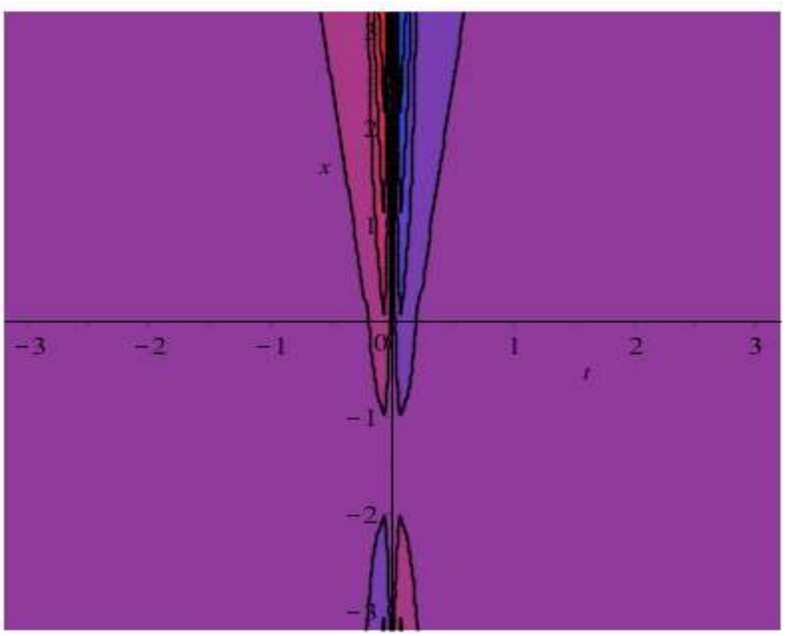


Figure 2

Density Plot of the Solution 2.18 with $\delta = \lambda = 1$



Traveling wave solution of the gkdv equation and its dynamical system analysis

The equation (1.2)'s traveling wave (periodic wave) is analyzed qualitatively for the initial instance in the literature. To investigate every potential traveling wave of the equation (1.2), we consider a linear combination of the symmetries:$X_s = X_1 + \lambda X_2$ where λ symbolizes the traveling wave's velocity.

By solving the characteristic equation

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$$\frac{dt}{1} = \frac{dx}{\lambda} = \frac{d\rho}{0} \tag{4.1}$$

The similarity variables are obtained for X_s given by

$$\zeta = x - \lambda t,\ \mathcal{U}(\zeta) = \rho \tag{4.2}$$

The GKDV equation (1.2) is reduced by these similarity variables to the third order ODE

$$-(\lambda + \delta)\mathcal{U}' + \frac{3}{2}\mathcal{U}\mathcal{U}' + \frac{1}{6}\mathcal{U}''' = 0 \tag{4.3}$$

When we integrate the transformed equation (4.26), in terms of z, we get

$$-(\lambda + \delta)\mathcal{U} + \frac{3}{4}\mathcal{U}^2 + \frac{1}{6}\mathcal{U}'' = k \tag{4.4}$$

where k can be any constant of integration. Utilizing the boundary requirements $\mathcal{U} \longrightarrow 0, \mathcal{U}' \longrightarrow 0, \mathcal{U}'' \longrightarrow 0$ as $\zeta \longrightarrow \pm\infty$, we have

$$-(\lambda + \delta)\mathcal{U} + \frac{3}{4}\mathcal{U}^2 + \frac{1}{6}\mathcal{U}'' = 0. \tag{4.5}$$

The system (4.5) is now reduced to the dynamical system

$$\begin{cases} \mathcal{U}' = v \\ v' = \alpha\mathcal{U} - \beta\mathcal{U}^2 \end{cases} \tag{4.6}$$

where $\alpha = 6(\lambda + \delta)$ and $\beta = \frac{9}{2}$. The system (4.6) is a dynamical system (Guckenheimer & Holmes, 1983); (Nieto & Torres, 2000); (Lakshmanan & Rajaseekar, 2003); (Saha, 2012); (Saha, 2017) with parameters δ and λ. Two equilibrium points exist for the dynamical system (4.29) at $E_0(U_0, \zeta_0) = E_0(0,0)$ and $E_1(U_1, \zeta_1) = E_1(\frac{\alpha}{\beta}, 0)$.

Periodic wave solution

Waves having a repeating pattern composed of cycles that repeat over a certain interval of time are known as periodic waves. This periodic waves, of the planar dynamical system (4.6) is a family of periodic trajectories centered around the equilibrium point $E_1(\mathcal{U}_1, \zeta_1) = E_1(\frac{\alpha}{\beta}, 0)$. Consequently, for the system (4.6) there are a number of periodic wave solutions that match the set of periodic trajectories around $E_0(\mathcal{U}_0, \zeta_0) = E_0(0,0)$ The distinction of the periodic waves of the GKdV equation (1.2) is presented numerically in Figure 3 for various velocities, λ = 1.0, 1.5 and 2.0 with fixed Coriolis parameter, $\delta = 0.4$ of the traveling wave. The periodic wave's amplitude increases and its width decreases as the velocity (λ) increases, as seen in Figure 3. Therefore, smoothness of the periodic wave in the equation (1.2) reduces as the nonlinear wave's velocity (λ) grows, thus, the periodic wave becomes spiky.

To examine the effect of the Coriolis force on the waves, we calculate the periodic waves of equation (1.2) numerically, as Figure 4 illustrates for various Coriolis parameter, δ = 0.1, 0.2 and 0.4 while velocity of the periodic waves, λ is kept at a constant, $\lambda = 1$. Figure 4 illustrates how the amplitude of the periodic wave shrinks and the width expands with an increase in the Coriolis parameter (δ). Therefore, as there is an increase of the Coriolis parameter (δ), the equation(1.2)'s periodic wave gets curvy and smooth.

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Figure 3

Distinction of Periodic Waves of (1.2) for various Velocities (λ)

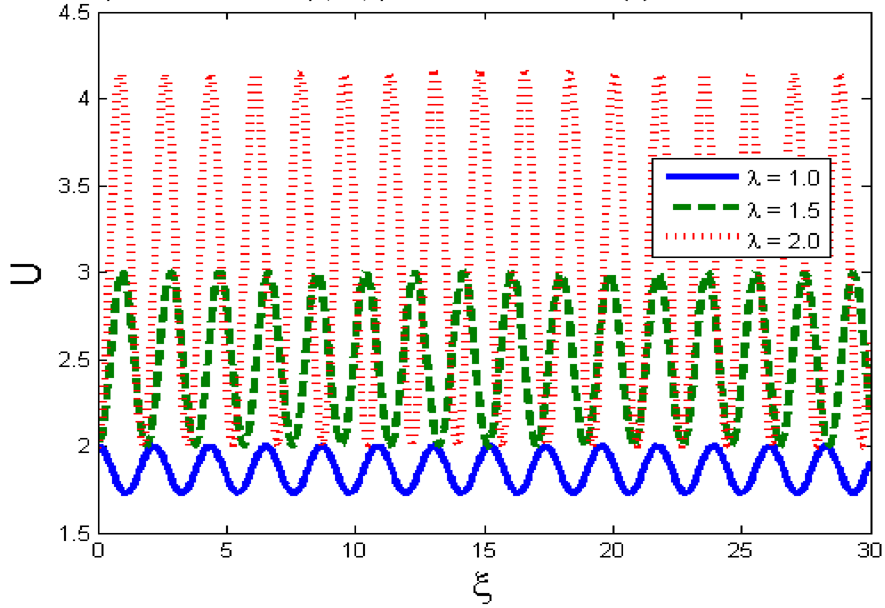
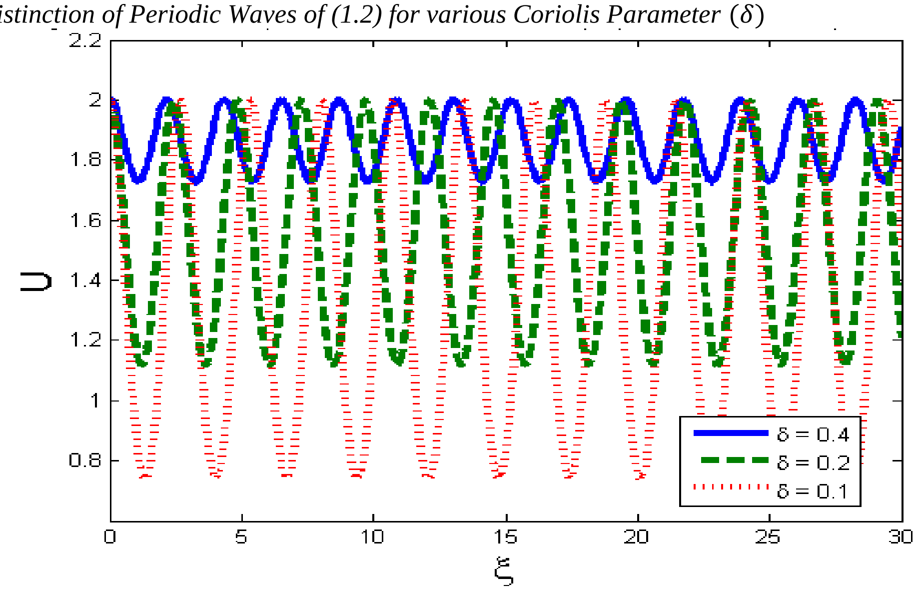


Figure 4

Distinction of Periodic Waves of (1.2) for various Coriolis Parameter (δ)



Discussion

From figure 3, we have disparity of periodic wave of the equation (1.2) for a range of values for the velocity (λ) of the travelling wave for a fixed value of the coriolis factor ($\delta = 0.4$). It is deduced from figure 3 that when the velocity (λ) increases, the amplitude of the periodic wave of equation (1.2) grows and then we have a decrease in the

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wave length. Thus, when velocity (λ) of the nonlinear wave grows, the smoothness of the periodic wave of the equation (1.2) decreases.

From figure 4, for a fixed velocity ($\lambda = 1$) of the travelling wave we have distinct periodic waves of the equation (1.2) for a range of values of coriolis factor ($\delta = 0.1, 0.2$ and 0.4). So, it is seen as the Coriolis factor (δ) increases, there is a decrease in the amplitude of the periodic wave and then the wave length expands. Thus when Coriolis factor (δ) increases, the periodic wave of the equation becomes curvy and smooth.

Conclusion

We use Lie symmetry analysis, dynamical system analysis, and conservation laws to examine the distinctive characteristics and attributes of the geophysical Korteweg-de Vries equation. With symmetry analysis, new exact solution is determined for the geophysical Korteweg – de Vries equation. This exact solution depicts similar structure of solitary wave as shown in Figure 1. Furthermore, we apply dynamical system analysis on the equation to examine the Coriolis effect on the free flow in oceans. The dynamical system analysis is examined in light of the traveling wave velocity and Coriolis factor. The study reveals that the velocity of the traveling wave, and Coriolis factor have significant effects on the transmission of the periodic wave solution of the GKDV equation. To illustrate how compelling our method is for solving nonlinear evolution equations, we use a graph to demonstrate our solutions. Furthermore, by applying multiplier's method, we compute conservation laws of the GKDV equation. The drawing of the wave solutions, symmetry analysis, dynamical system analysis and enumeration of the conserved quantities of the GKDV equation by us are relatively new and recent discoveries (Figures 1-4).

Recommendation

The findings of this study could be applied to the investigation of disruptions or wave propagation issues in oceanic flows in the equatorial region. The method used here is straightforward and conventional, so it serves as motivation to extend the method to some other nonlinear evolution models, like the generalized Hirota-Satsuma coupled KdV system (Lu et al., 2007) the Davey-Stewartson system, the coupled Kadomtsev-Petviashvili system, and so forth.

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