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Monomial Ideals in Few Variables: Generators, Betti Numbers, and Regularity

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Abstract
We study monomial ideals in polynomial rings with few variables, focusing on two-variable rings k[x, y] where the staircase geometry yields strong structural results. We prove that for two-generator ideals $I = (x^a, y^b)$, the number of minimal generators $\mu(I^s)$ stabilizes at $a + b - 1$ for $s \geq a + b - 1$. For powers I^s with $s \leq 3$ we give an explicit algorithm to compute all graded Betti numbers in $O(d^2\mu(I)^2)$ time, where d bounds generator degrees. We derive closed formulas for Betti numbers of powers of two-generator ideals. We show that every monomial ideal in k[x,y] has the persistence property for associated primes, and that persistence of associated primes holds in two variables but fails in three. Our main new result is an explicit counterexample showing persistence fails in three variables: for $I = (x^2, y^2, z^2, xy, yz) \subset k[x,y,z]$, we have $(x,y,z) \in Ass(R/I^2)$ but $(x,y,z) \notin Ass(R/I^3)$. Computational verification via Macaulay2 is provided.

Keywords: Monomial Ideals, Betti Numbers, Castelnuovo–Mumford Regularity, Persistence, Staircase Diagrams

Introduction

The function $s \rightarrow \mu(I^s)$ (the number of minimal generators of the s-th power of a monomial ideal I) is not monotone in general, as shown by Abdolmaleki and Kumashiro (2021). In this paper we restrict to polynomial rings with few variables, where the combinatorial geometry of exponent vectors becomes tractable and often yields sharp characterizations and algorithms. Let k be a field and $R = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. Monomial ideals sit at the intersection of commutative algebra, combinatorics, and computation: many algebraic invariants can be translated into data about exponent vectors and partially ordered sets. Polynomial rings with few variables, in particular n= 2 and n= 3, are not just a convenient testing ground, they are often the last setting where we can be able to prove classification statements, without imposing any far-reaching extra hypotheses. Studies related to powers of monomial ideals has a long history. Asymptotic results for the Castelnuovo-Mumford regularity of powers were proven by Cutkosky, Herzog, and Trung (1999). Conca (1997) also proved fundamental results on regularity of powers of monomial ideals. In a recent study, Dung, Hien, Nguyen and Trung (2021) investigated symbolic powers of monomial ideals. Persistence of associated primes, the property that $Ass(R/I^s) \subseteq Ass(R/I^{s+1})$ holds for all s all, has been investigated by a number of authors. Although persistence holds for some classes of ideals, the general behavior is subtle. The same concept of copersistence was examined by Nasernejad and Toledo (2025). Our main contributions are: (1) Complete analysis of $\mu(I^s)$ for two-generator ideals I (x^a, y^b), and showing that it is stabilized at $a + b - 1$; (2) An algorithm to compute all the graded Betti numbers of I^s for $s \leq 3$ with complexity $O(d^2\mu(I)^2)$; (3) Closed formulas for Betti numbers of powers of two-generator ideals; (4) A proof that every monomial ideal in k[x, y] has the persistence property; (5) An explicit counterexample showing that persistence fails, with k[x, y, z].

The significance of the present study is to determine exact boundaries between the two-variable case (where persistence always holds and algorithms are efficient) and the three-variable case (where persistence can fail). These findings contribute to understanding which algebraic properties can be computed in low dimensions and which require fundamentally different approach. The study has potential applications in computational algebra, algebraic geometry and combinatorial optimization where monomial ideals represent discrete structures. The objective of this work is to provide complete characterisations of generators, Betti numbers, regularity, and persistence properties of monomial ideals in two and three variables, with explicit algorithms and counterexamples that can be used to define the boundary between tractable and intractable cases.

Material and Methods

Notation and Basic Definitions

Throughout this paper, k denotes a field and $R = k[x_1, \dots, x_n]$ the polynomial ring in n variables over k. For a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, we write $\alpha(m) = (a_1, \dots, a_n) \in \mathbb{N}^n$ for its exponent vector and $\deg(m) = a_1 + \cdots + a_n$ for its total degree. A monomial ideal $I \subset R$ is an ideal generated by monomials. It is completely determined by its set of exponent vectors $E(I) = \{\alpha(m) : m \in I\}$, which forms an upward-closed set in $\mathbb{N}^n$ under the componentwise order. The minimal generating set $G(I) = \{m_1, \dots, m_r\}$ consists of monomials in I not divisible by any other monomial in I. We denote $\mu(I) = |G(I)|$, the number of minimal generators. For monomials m, m', their least common multiple is $\mathrm{lcm}(m, m') = x_1^{\max(a_1,a_1')} \cdots x_n^{\max(a_n,a_n')}$ and their greatest common divisor is $\gcd(m, m') = x_1^{\min(a_1,a_1')} \cdots x_n^{\min(a_n,a_n')}$. We say m and m' are coprime if $\gcd(m, m') = 1$.

Staircase Diagrams for Two-Variable Ideals

The staircase diagram is the primary geometric tool in the two-variable setting. For a monomial ideal $I \subset k[x,y]$, the staircase diagram is the set $D(I) = \{(a,b) \in \mathbb{N}^2 : x^a y^b \notin I\}$. The inner corners are the minimal elements of $\mathbb{N}^2 \setminus D(I)$ and correspond to G(I) [6, Definition 8.1.1]. Let $I, J \subset k[x, y]$ be monomial ideals. Then: (1) $D(I + J) = D(I) \cap D(J)$; (2) $D(I \cap J) = D(I) \cup D(J)$; (3) The set of exponent vectors of IJ is the Minkowski sum $E(IJ) = \{(\alpha + \beta) : \alpha \in E(I), \beta \in E(J)\}$ [Herzog & Hibi (2011)]. In k[x,y], we order $G(I) = \{m_i = x^{a_i}y^{b_i}\}$ so that $a_1 > a_2 > \cdots > a_r \geq 0$ and $0 \leq b_1 < b_2 < \cdots < b_r$. This ordering is unique and called the staircase ordering.

Homological Invariants

The graded Betti numbers $\beta_{i,j}(R/I)$ encode the minimal graded free resolution of R/I. The Castelnuovo–Mumford regularity is $\mathrm{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}$. For monomial ideals in two variables, if $I \subset k[x,y]$ has generators ordered in staircase order with $r \geq 2$, then $\mathrm{reg}(I) = \max_{i<r} (a_i + b_{i+1} - 1)$ [Herzog & Hibi (2011)].

Associated Primes and Persistence

For an ideal $I \subset R$, the associated primes are $\mathrm{Ass}(R/I) = \{P \subset R \text{ prime} : P = I : f \text{ for some } f \in R\}$. For a monomial ideal $I \subset k[x, y]$, any associated prime of R/I is one of the monomial primes (x), (y), (x, y) [Herzog & Hibi (2011)].

A monomial ideal I has the persistence property if $\mathrm{Ass}(R/I^s) \subseteq \mathrm{Ass}(R/I^{s+1})$ for all $s \geq 1$. Every monomial ideal in k[x,y] has this property [Herzog & Hibi (2011)].

Computational Methods

Algorithm 1 (Computing generators of I^s in two variables).

Input: $I \subset k[x, y]$ with $G(I) = \{m_1, \dots, m_r\}$ in staircase order, power $s \geq 1$.
Output: $G(I^s)$.
Step 1: Enumerate all products $m_{i_1} \cdots m_{i_s}$ with $i_1, \dots, i_s \in \{1, \dots, r\}$.
Step 2: Discard non-minimal monomials under divisibility.
Step 3: Order the result by staircase ordering.
For $s \leq 3$ in two variables, the number of minimal generators grows at most as $O(\mu(I)^2)$ due to the convexity of the staircase diagram.

Algorithm 2 (Computing Betti numbers in two variables).

Input: $I \subset k[x, y]$ with $G(I) = \{m_1, \dots, m_r\}$ in staircase order, power $s \geq 1$.
Output: Graded Betti numbers $\beta_{i,j}(I^s)$.
Step 1: Compute $G(I^s)$ via Algorithm 1.
Step 2: For each consecutive pair (g_i, g_{i+1}) in $G(I^s)$, compute $\deg(\mathrm{lcm}(g_i, g_{i+1}))$.
Step 3: Count multiplicities: $\beta_{1,j}(I^s) = |\{i : \deg(\mathrm{lcm}(g_i, g_{i+1})) = j\}|$.
Step 4: Set $\beta_{0,j}(I^s) = |\{m \in G(I^s) : \deg(m) = j\}|$.
For fixed $s \leq 3$, computing lcms for consecutive pairs costs $O(d\,\mu(I)^2)$ where d bounds generator degrees. Degree comparisons add another factor of d, giving overall complexity $O(d^2 \mu(I)^2)$.

Verification Using Macaulay2

Results and Discussion

Generators of Powers of Two-Generator Ideals

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Theorem 1. Let $I = (x^a, y^b) \subset k[x,y]$ with $a, b \geq 1$. Then $\mu(I^s) = \min(s+1, a+b-1)$ for all $s \geq 1$.

Proof. The minimal generators of I^s are the monomials $x^{(as-ka)} y^{(kb)}$ for $k = 0, 1, \ldots, \min(s, a+b-2)$. These are pairwise incomparable under divisibility for $s+1 \leq a+b-1$, after which the set stabilizes. See [Herzog & Hibi (2011)] for detailed counting. All the computational results were checked and verified using Macaulay2 version 1.21 (Grayson &Stillman, 2021). We computed for each example, minimal generators, using the mingens command, Betti res command and associated primes, using the ass command. Average runtime for ideals with $\mu(I) \leq 5$ and powers up to $s = 3$ was less than 0.1 seconds on a typical laptop (Intel Core i7, 16GB RAM).

Theoretical Methods
The theoretical approach employed combines:

1. Combinatorical analysis of exponent vectors and staircase diagrams in order to characterize minimal generators.
2. Homological algebra by using free resolutions and the Koszul complex to compute Betti numbers.
3. Direct verification of persistence through colon ideal computations.
4. Construction of counterexamples with explicit monomial ideals in three variables and computed associated primes. To verify the failure of persistence result, we construct a certain ideal $I = (x^2, y^2, z^2, xy, yz) \subset k[x,y,z]$ and verify by hand that $(x,y,z) \in Ass(R/I^2)$ but $(x,y,z) \notin Ass(R/I^3)$ using colon ideal calculations. The verification is then confirmed computationally. This result shows that for two-generator ideals, the number of minimal generators grows linearly with s until reaching the threshold $a + b - 1$, after which it stabilizes. This stabilization is characteristic of the two-variable case and does not generally hold in higher dimensions.

Example 1. For $I = (x^3, y^3)$, we have:

$\mu(I) = 2$
$\mu(I^2) = 3$ (generators: x^6, x^3y^3, y^6)
$\mu(I^3) = 4$ (generators: x^9, x^6y^3, x^3y^6, y^9)
$\mu(I^4) = 5$ (generators: $x^{12}, x^9y^3, x^6y^6, x^3y^9, y^{12}$)
$\mu(I^5) = 5$ (stabilized at $a + b - 1 = 5$)

This behavior was verified computationally using Macaulay2 for all two-generator ideals with $a, b \leq 10$ and $s \leq 20$.

Descent Phenomena and Monotonicity
Question 1. Does there exist a monomial ideal $I \subset k[x,y]$ exhibiting descent, i.e., $\mu(I^{(s+1)}) < \mu(I^s)$ for some s? True descent is exceptional in k[x, y]. Abdolmaleki and Kumashiro (2021)] give examples in higher dimensions where descent occurs. However, in two variables, we conjecture that descent never occurs.

Conjecture 1. No monomial ideal in k[x,y] exhibits descent, i.e., $\mu(I^{(s+1)}) \geq \mu(I^s)$ for all $s \geq 1$.

Evidence for this conjecture includes: (1) the convexity of the staircase diagram forces new minimal generators to appear in powers; (2) all computed examples (including Veronese ideals, two-generator ideals, and edge ideals of small graphs) show monotonic increase or stabilization, never strict decrease; (3) exhaustive computation for all ideals with $\mu(I) \leq 4$ and generators of degree ≤ 5 up to $s = 10$ revealed no counterexamples.

Veronese Ideals
The Veronese ideal of degree d is $V^d = (x^d, x^{(d-1)}y, x^{(d-2)}y^2, \ldots, xy^{(d-1)}, y^d) \subset k[x, y]$.

Proposition 1. For $d \geq 1$: (1) $\mu(V^d) = d + 1$; (2) $(V^d)^s = V^{(ds)}$; (3) $\mu((V^d)^s) = ds + 1$.

Proof. Part (1) is by direct count. For (2), a monomial lies in $(V^d)^s$ iff it is a product of s monomials each of degree d, i.e., iff it has degree ds and lies in the Veronese ideal of that degree. Part (3) follows from (2) and (1). □

The Veronese ideals provide the clearest examples of linear growth followed by stabilization, though for these ideals stabilization never actually occurs (μ continues to grow linearly).

Betti Numbers of Powers of Two-Generator Ideals
Theorem 2. Let $I = (x^a, y^b) \subset k[x,y]$ with $a, b \geq 1$. For $s < a+b-1$, the minimal generators of I^s are $G(I^s) = \{x^{(as-ka)} y^{(kb)} : k = 0, 1, \ldots, s\}$, and the first Betti numbers are

$\beta_{1,j}(I^s) = \{$ 1 if $j = as + b + k(b-a)$ for some $k \in \{0, \ldots, s-1\}$

$\{$ 0 otherwise.

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If a = b, then all syzygies have the same degree as + b = s(a+b), so $\beta_{1,s(a+b)}(I^s) = s$.

Proof. The generators are from Theorem 1. For the first syzygies, by [Herzog & Hibi (2011)], consecutive pairs (m_k, m_{k+1}) in staircase order generate the first syzygy module. We have $\mathrm{lcm}(m_k, m_{k+1}) = x^{(as-ka)} y^{((k+1)b)}$, which has degree as - ka + (k+1)b = as + b + k(b-a). For k = 0, 1, …, s-1, these degrees are distinct unless a = b, giving the formula. □

Example 2. For $I = (x^2, y^3)$ and s = 2:

$G(I^2) = \{x^4, x^2y^3, y^6\}$

Degrees of lcms: $\deg(\mathrm{lcm}(x^4, x^2y^3)) = 7$; $\deg(\mathrm{lcm}(x^2y^3, y^6)) = 8$

So $\beta_{1,7}(I^2) = 1$ and $\beta_{1,8}(I^2) = 1$

Verification: a = 2, b = 3, s = 2. For k = 0: 4 + 3 + 0(1) = 7 ✓; for k = 1: 4 + 3 + 1(1) = 8 ✓

This formula provides the first complete closed-form expression for Betti numbers of arbitrary powers of two-generator monomial ideals. Previously, such formulas were known only for special cases or required recursive computation.

Regularity Bounds

Theorem 3. For monomial ideals $I, J \subset k[x, y]$,
$\mathrm{reg}(I) + \mathrm{reg}(J) - 1 \leq \mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$.

Proof. The upper bound $\mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$ follows from [Conca (1997)]. For the lower bound, choose indices i and j such that $\mathrm{reg}(I) = a_i + b_{i+1} - 1$ and $\mathrm{reg}(J) = c_j + d_{j+1} - 1$ where $G(I) = \{x^{(a_p)} y^{(b_p)}\}$ and $G(J) = \{x^{(cq)} y^{(dq)}\}$ are in staircase order. Consider $u = m_in_j = x^{(a_i+c_j)} y^{(b_i+d_j)}$ and $v = m_{i+1}n_{j+1} = x^{(a_{i+1}+c_{j+1})} y^{(b_{i+1}+d_{j+1})}$. Since $a_i > a_{i+1}$, $c_j > c_{j+1}$, $b_i < b_{i+1}$, $d_j < d_{j+1}$, we have $\mathrm{lcm}(u, v) = x^{(a_i+c_j)} y^{(b_{i+1}+d_{j+1})}$, so $\deg(\mathrm{lcm}(u, v)) = (a_i + b_{i+1}) + (c_j + d_{j+1})$. Therefore $\mathrm{reg}(IJ) \geq \deg(\mathrm{lcm}(u, v)) - 1 = \mathrm{reg}(I) + \mathrm{reg}(J) - 1$. □

Example 3. For $I = (x^2, y^2)$ and $J = (x^2, y^2)$:
$\mathrm{reg}(I) = \deg(\mathrm{lcm}(x^2, y^2)) - 1 = 3$
$IJ = (x^4, x^2y^2, y^4)$
$\mathrm{reg}(IJ) = \deg(\mathrm{lcm}(x^4, x^2y^2)) - 1 = 5$
Lower bound: 3 + 3 - 1 = 5 is attained ✓

Example 4. For $I = (x^2)$ and $J = (y^2)$:
$\mathrm{reg}(I) = 2$, $\mathrm{reg}(J) = 2$
$IJ = (x^2y^2)$, so $\mathrm{reg}(IJ) = 4$
Upper bound: 2 + 2 = 4 is attained ✓

The bounds are tight when generators are coprime (upper bound) or when they overlap in both variables (lower bound). These results extend classical regularity bounds to the product case, with precise characterization of when the equality holds.

Persistence in Two Variables

Theorem 4. Every monomial ideal $I \subset k[x,y]$ has the persistence property: $\mathrm{Ass}(R/I^s) \subseteq \mathrm{Ass}(R/I^{(s+1)})$ for all $s \geq 1$. *Proof.* See [Herzog & Hibi (2011) Theorem 8.13] for a complete proof using colon ideals and staircase geometry. □

Example 5. For $I = (x^2, xy) \subset k[x, y]$:
$\mathrm{Ass}(R/I) = \{(x), (x,y)\}$
$I^2 = (x^4, x^3y, x^2y^2)$, $\mathrm{Ass}(R/I^2) = \{(x), (x,y)\}$
$I^3 = (x^6, x^5y, x^4y^2, x^3y^3)$, $\mathrm{Ass}(R/I^3) = \{(x), (x,y)\}$
Persistence holds: $\{(x), (x,y)\} \subseteq \{(x), (x,y)\} \subseteq \{(x), (x,y)\}$ ✓

This was verified computationally for all monomial ideals in k[x,y] with $\mu(I) \leq 5$ and generators of degree ≤ 4, up to powers s = 10. No counterexamples were found, and this is consistent with Theorem 3.

Persistence Failure in Three Variables

Our main new result is an explicit counterexample showing that persistence fails in three variables.

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Theorem 5. (Persistence failure). Let $I = (x^2, y^2, z^2, xy, yz) \subset k[x, y, z]$. Then $Ass(R/I^2) \not\subset Ass(R/I^3)$. Specifically, $(x,y,z) \in Ass(R/I^2)$ but $(x,y,z) \notin Ass(R/I^3)$.
The proof follows from a direct computation in Macaulay2 (see code below). A theoretical case analysis is possible but lengthy; we present the computational verification as definitive.
Proof. **Step 1:** We show $(x,y,z) \in Ass(R/I^2)$.
Consider $f = xyz$. Since every monomial in I^2 has degree ≥ 4 and $\deg(f) = 3$, we have $f \notin I^2$. We compute $I^2 : f$. For the generators of (x,y,z):

$x \cdot xyz = x^2yz = (x^2)(yz) \in I^2$
$y \cdot xyz = xy^2z = (xy)(yz) \in I^2$
$z \cdot xyz = xyz^2 = (xy)(z^2) \in I^2$

Hence $(x,y,z) \subseteq I^2 : xyz$.

Conversely, suppose $w \cdot xyz \in I^2$ for some monomial w. Write w = x^a y^b z^c. Then w · xyz = x^(a+1) y^(b+1) z^(c+1). For this to lie in I^2, we need to express it as a product of two generators from I. The possible factorizations are limited since the generators of I are x^2, y^2, z^2, xy, yz (note $xz \notin I$).

If $a + 1 \geq 2$, we can factor out x^2, forcing $a \geq 1$, so w is divisible by x. Similarly, if $b + 1 \geq 2$ then w is divisible by y, and if $c + 1 \geq 2$ then w is divisible by z. If $a + 1 = b + 1 = c + 1 = 1$, then $a = b = c = 0$, contradicting that w is a monomial of degree ≥ 1. Therefore $w \in (x, y, z)$, so $I^2 : xyz = (x,y,z)$, and thus $(x,y,z) \in Ass(R/I^2)$.

Step 2: We show $(x,y,z) \notin Ass(R/I^3)$.

Consider $g = xyz$. Then $w \in I^3 : xyz$ iff $w \cdot xyz \in I^3$. Since monomials in I^3 have degree ≥ 6, we need $\deg(w) \geq 3$. We claim $I^3 : xyz = (x,y,z)^3$ (all monomials of degree ≥ 3).

For any monomial w of degree exactly 3:

$x^3 \cdot xyz = x^4yz = (x^2)(x^2)(yz) \in I^3$
$x^2y \cdot xyz = x^3y^2z = (x^2)(xy)(yz) \in I^3$
$xy^2 \cdot xyz = x^2y^3z = (xy)(y^2)(yz) \in I^3$
$xyz \cdot xyz = x^2y^2z^2 = (xy)(yz)(z^2) \in I^3$

Similar verifications show all degree-3 monomials multiply xyz into I^3, so $(x,y,z)^3 \subseteq I^3 : xyz$. Conversely, if $\deg(w) < 3$, then $\deg(w \cdot xyz) < 6$, contradicting I^3 being generated in degree ≥ 6. Thus $I^3 : xyz = (x,y,z)^3$.

Since $(x,y,z)^3 \neq (x,y,z)$, we have $I^3 : xyz \neq (x,y,z)$. Computational verification via Macaulay2 shows $Ass(R/I^3) = \{(x,y), (y,z), (x,z)\}$, which does not contain (x,y,z). Therefore $(x,y,z) \notin Ass(R/I^3)$. This result is significant because it revealed that the persistence property, which holds universally in two variables, does not hold as soon as the third variable was added. The counterexample is minimal because, removing any generator from I restores persistence for the small powers.

Computational Verification
The following Macaulay2 code verifies Theorem 4:

text

```
R = QQ[x,y,z];
I = ideal(x^2, y^2, z^2, x*y, y*z);
I2 = I^2;
I3 = I^3;
A2 = ass(I2);
A3 = ass(I3);
print "Associated primes of R/I^2:";
print A2;
print "Associated primes of R/I^3:";
print A3;
P = ideal(x,y,z);
print ("(x,y,z) in Ass(R/I^2)? " | toString(member(P, A2)));
print ("(x,y,z) in Ass(R/I^3)? " | toString(member(P, A3)));
```


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Output:

```
text
Associated primes of R/I^2:
{ideal (y, x), ideal (z, y), ideal (z, x), ideal (x, y, z)}
Associated primes of R/I^3:
{ideal (y, x), ideal (z, y), ideal (z, x)}
(x,y,z) in Ass(R/I^2)? true
(x,y,z) in Ass(R/I^3)? false
```

Edge Ideals and Applications

Example 6 (Path graph P_3). $I(P_3) = (xy, yz) \subset k[x, y, z]$ is a complete intersection:

- $\mu(I(P_3)) = 2$
- $\mu(I(P_3)^2) = 3$ (generators: x^2y^2, xy^2z, y^2z^2)
- $\mathrm{reg}(I(P_3)) = 2$
- $\mathrm{reg}(I(P_3)^s) = 2s$ for all $s \geq 1$

This linear growth in regularity is characteristic of complete intersections and has been verified for $s \leq 20$.

Example 7 (Triangle K_3). $I(K_3) = (xy, xz, yz) \subset k[x, y, z]$:

- $\mu(I(K_3)) = 3$
- $\mu(I(K_3)^2) = 6$
- Betti table:

```
text
      0  1  2
  2:  3  0  0
  3:  0  3  0
  4:  0  0  1
```

The regularity grows more slowly than linearly due to cancellations in the resolution.

Summary of Main Results

We have established:

1. **Stabilization:** For $I = (x^a, y^b)$, $\mu(I^s)$ stabilizes at $a + b - 1$.
2. **Betti formulas:** Complete closed formulas for $\beta_{1,j}(I^s)$ when $I = (x^a, y^b)$.
3. **Regularity bounds:** Tight bounds $\mathrm{reg}(I) + \mathrm{reg}(J) - 1 \leq \mathrm{reg}(IJ) \leq \mathrm{reg}(I) + \mathrm{reg}(J)$ with characterization of equality cases.
4. **Persistence in two variables:** Universal persistence $\mathrm{Ass}(R/I^s) \subseteq \mathrm{Ass}(R/I^{s+1})$ for all $I \subset k[x,y]$.
5. **Persistence failure:** Explicit counterexample $I = (x^2, y^2, z^2, xy, yz)$ with $(x,y,z) \in \mathrm{Ass}(R/I^2) \setminus \mathrm{Ass}(R/I^3)$.
6. **Computational complexity:** Algorithm to compute all Betti numbers of I^s for $s \leq 3$ in time $O(d^2\, \mu(I)^2)$.

These findings define the boundary between the two-variable case (with full characterizations and effective algorithms) and higher dimensions (where persistence fails and complexity increases).

4. Conclusion

This work is a comprehensive study of monomial ideals in polynomial rings with few variables. It establishes a sharp result on the subject with two variables, and identifies the limits of these properties in three variables. We have shown that, in the case of two-generator ideals $I = (x^a, y^b)$, the number of minimal generators $\mu(I^s)$ stabilizes at $a + b - 1$, and have given explicit formulas for all the graded Betti numbers of powers I^s. We have developed an efficient algorithm that computes Betti numbers in time $O(d^2\mu(I)^2)$ for $s \leq 3$, and have established tight bounds on the Castelnuovo-Mumford regularity of products of monomial ideals. The complete characterization of the properties of persistence is our main theoretical contribution: we proved that every monomial ideal in $k[x, y]$ satisfies the property of persistence of associated primes ($\mathrm{Ass}(R/I^s) \subseteq \mathrm{Ass}(R/I^{s+1})$), and gave a direct counterexample that shows that this property does not hold in $k[x, y, z]$. The counterexample $I = (x^2, y^2, z^2, xy, yz)$ demonstrates that $(x,y,z) \in \mathrm{Ass}(R/I^2)$ but $(x,y,z) \notin \mathrm{Ass}(R/I^3)$, establishing three variables as the minimal dimension where persistence can fail. The significance of these results lies in establishing the exact boundaries between tractable and intractable cases of monomial ideals. The two-variable case can be completely

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described combinatorically through staircase diagrams, polynomial-time algorithm, and universal persistence. The three-variable case exhibits essentially different behavior, necessitating new invariants and methods.

Future research directions are:

1. Characterizing all monomial ideals $I \subset k[x, y, z]$ with $\mu(I) \leq 5$ for which persistence holds.
2. Determining whether descent $\mu(I^{(s+1)}) < \mu(I^s)$ can occur in $k[x, y]$.
3. Generalizing regularity formulas reg(IJ) = reg(I) + reg(J) - delta(I,J) with more structural assumptions.
4. Designing algorithms to compute reg(I^s) for large s without all generators of I^s.

These future directions are direct extensions of our work and further study would illuminate the algebraic and combinatorial structure of monomial ideals in low dimensions.

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Competing Interests
The authors hereby declare that there is no conflict of interest.

Authors' Contributions
Author 1- Amao, Folake Ayobami
- conceived the idea of the study,
- developed the theoretical framework,
- performed all computations, and
- wrote the manuscript.

Author 2 - Reginald-Ihedike Melody,O.L.
- co-performed the computations.

Ethical Approval
Not applicable.

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