Faculty of Natural and Applied Sciences Journal of Mathematical and Statistical Computing
Print ISSN: 3026-8605, e-ISSN 3043-632X
www.fnasjournals.com
Volume 2; Issue 1; September 2024; Page No. 39-43.

The Commutative Properties of Multigroup

[1]Michael, B.D., [2]Nwagrabe, E., & [3]Awolola, J.A.
[1]Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara Kogi State, Nigeria
[2]Department of Mathematics/Statistics, Ignatius Ajuru University of Education Port Harcourt, Nigeria
[3]Department of Mathematics, Joseph Sarwuan Tarka University, Makurdi Benue State, Nigeria

***Corresponding author email:** michaeldb@custech.edu.ng

Abstract
Multigroups is a non - classical algebraic structure which generalize group theory over a multiset which has been explored by many researchers and some results are established which are analogue to classical group theory. The notion of center, centralizer and normalizer in multigroup is studied in this paper to capture the notion of multiset since multigroup is a generalization of group theory and it will enhance further study on multigroup theory. It was established that center and normalizer of multigroup is a normal submultigroups and is the given multigroup if it is commutative.

Keywords: Multigroup, Submultigroups, Center, Centralizer and Normalizer

Introduction
The concept of multigroup over was established by (Nazmul et al., 2013) as an algebraic structure over a multiset that generalized the notion of group. The perspective is consistent with other non-classical groups (Rosenfeld, 1971), (Shinoj et al., 2015), (Shinoj & Sunil, 2015) and (Nazmul et al., 2011). Researchers in (Awolola & Ibrahim, 2016), (Awolola & Ejegwa, 2017), (Ejegwa, 2017), (Michael & Ibrahim, 2023) and many others have studied many properties of multigroup and its commutative nature. Ejegwa and Ibrahim, (2017) normalizer of submultigroups but it was proven to be a subgroup. In 2020, Ejegwa and Ibrahim, defined center of multigroup as a subgroup of the root set also and proved that center of multigroup is a group if the multigroup is commutative and, Gyam and Ogwola (2023) defined center of multigroup as a subgroup of the root set which contradict the generalization of group. In this paper, we propose the notion of center of multigroup, centralizer of submultiset and normalizer of submultigroups to capture multiplicity of objects in a set unlike what the earlier researchers proposed.

Preliminary
This section presents a fundamental definition on multigroups that will be used in the subsequent section of this paper.

Definition 2.1 (Singh et al., 2007) Let X be a set. A multiset A drawn from X is presented by a count function C_G defined as $C_A : X \longrightarrow D = \{0, 1, 2, 3, \dots\}$. For each $x \in X, C_A(x)$ denotes the number of occurrences of the element x in the multiset A drawn from $X = \{x_1, x_2, x_3, \dots, x_n\}$ will be as $A = [x_1, x_2, x_3, \dots, x_n]_{m_1, m_2, m_3, \dots, m_n}$ such that x_i appears m_i times, $i = 1, 2, \dots., n$ in the multisets A.

Definition 2.2 (Nazmul et al., 2013) Let X be a group. A multiset A over X is called a multigroup over X if the count function C_G satisfies the following conditions;

i. $C_G(xy) \geq C_G(x) \wedge C_G(y),\ \forall x, y \in X,$
ii. $C_G(x^{-1}) \geq C_G(x),\ \forall x \in X.$

Definition 2.3 (Nazmul et al., 2013) Let X be a group and $G \in MG(X)$, then G is said to be commutative or abelian if $C_G(xy) = C_G(yx), x, y \in X$.
Definition 2.4 (Nazmul et al., 2013) Let X be a group and $G \in MG(X)$, the set $G_* \coloneqq \{x \in X : C_G(x) \geq 0\}$ is called the root set or support of multigroup G.

Definition 2.5 (Nazmul et al., 2013) Let X be a group and $G \in MG(X)$, then G inverse, $G^{-1} \in MG(X)$ is defined by $C_{G^{-1}}(x) = C_G(x^{-1}), \forall\, x \in X$.

Definition 2.6 (Ejegwa and Ibrahim, 2020) Let $G \in MG(X)$. A submultiset H of G is called a submultigroup if H form a multigroup and is denoted as $H \sqsubseteq G$. A submultigroup H of G is called a proper submultigroup denoted by $H \sqsubset G$, if $H \sqsubseteq G$ and $G \neq H$.
Definition 2.7 (Nazmul et al., 2013) A submultigroup, H of a multigroup $G \in MG(X)$ is normal submultigroup if and only if $h \in H_* : C_H(xhx^{-1}) \geq C_H(h), \forall x \in G_*$.

Remark In general, $H \sqsubseteq G$ if $y \in H_*$ then $C_H(xy^{-1}) \geq C_H(x) \wedge C_H(y) \;\forall\, x, \in G_*$.

Definition 2.8 (Ejegwa and Ibrahim, 2020) Center of Multigroup
Let G be a multigroup over a group X, then the center of G is defined as

$$C_G(x) = \{x \in X \mid C_G([x, y]) = C_G(e), \forall y \in X\}$$

Definition 2.9 (Ejegwa &Ibrahim, 2020) Let $B \in MG(X)$ and A be a submultiset of B, then the centralizer of a submultiset A of B is the set

$$Z(A) = \{x \in X \mid C_G(xy) = C_G(yx) \text{ and } C_G(xyz) = C_G(yxz), \forall y, z \in X\}$$

Definition 2.10 (Ejegwa and Ibrahim, 2020) Let G be a group and A be a subgroup of G, then normalizer of A in G is $N_G(A) \coloneqq \{x \in G \mid xy = yx, \forall y \in A\}$

Proposition 2.11 (Gyam & Ogwola, 2023**)** Let $G \in MG(X)$. Then the centralizer of G denoted by $C(M) = \{\mathrm{x} \in X : C_M(xy) = C_M(yx), \forall\, y \in X\}$.

Results
Definition 3.1 Let G be a multigroup over a group X then the center of G, denoted by $C(G)$ is the submultiset of G generated by

$$C(G) \coloneqq \begin{cases} C_G(x), \text{if } C_G(xy) = C_G(yx), \forall y \in X \\ \quad 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Example 3.2 Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ be a group under matrix multiplication for

$$\rho_1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \rho_2 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \rho_3 = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}, \rho_4 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix},$$
$$\rho_5 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \rho_6 = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \rho_7 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \rho_8 = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}.$$

Therefore $G = [{\rho_1}^7, {\rho_2}^2, {\rho_3}^4, {\rho_4}^2, {\rho_5}^2, {\rho_6}^2, {\rho_7}^4, {\rho_8}^5] \in MG(X)$, and $C(G) = [{\rho_1}^{10}, {\rho_3}^7]$

Remark 3.3 obviously, $C_{C(G)}(x) > 0$ since $\exists x \in C(G)_* \mid C_G(xx^{-1}) = C_G(e), \forall x \in X$.

Theorem 3.4 Let $G \in MG(X)$, then $C(A) \sqsubseteq G$.

Proof Let G be a multigroup over X such that $C_{C(G)}(x) > 0$ since $C_{C(G)}(x) \leq C_G(x), \;\forall x \in C(G)_*$
For any $x \in X$, suppose $C_{C(G)}(x) = C_G(x)$ when $C_G(xy) = C_G(yx), \forall y \in X$ then
$\exists z \in X, \mid C_{C(G)}(xy) = C_G(z)$ and $C_{C(G)}(yx) = C_G(z), \forall x, y \in X$
Consequently, $C_{C(G)}(xy) = C_{C(G)}(yx), \forall y \in X$
Hence, $C(G)$ is a multiset and then for any $x, y \in X, C_{C(G)}(xy) = C_{C(G)}(xy), \forall y \in (C(G))_*$
$\Longrightarrow \exists e \in X \mid C_{C(G)}(xx^{-1}) = C_{C(G)}(e), \forall y \in X$
$\Longrightarrow C_{C(G)}(x) = C_{C(G)}(x^{-1}) > 0, \forall x \in X$
Clearly, $C_{C(G)}(x) = 0 \iff C_{C(G)}(x^{-1}) = 0$
Therefore it sufficient to show that, $C_{C(G)}(xy) \geq C_{C(G)}(x) \wedge C_{C(G)}(y^{-1})$ for $x, y \in X$.

Cite this article as:
Michael, B.D., Nwagrabe, E., & Awolola, J.A. (2024). The commutative properties of multigroup. *FNAS Journal of Mathematical and Statistical Computing*, *2*(1), 39-43.

From **Definition 3 .1** for any $x \in X$, $C_{C(G)}(x) \leq C_G(x)$, if $C_G(xy) = C_G(yx), \forall y \in X$
Suppose $C_{C(G)}(x) = C_G(ab) = C_G(ba), a, b \in X$ and $C_{C(G)}(y) = C_{C(G)}(cd) = C_{C(G)}(dc)$
$C_{C(G)}(xy) \geq \{C_G(a) \wedge C_G(b)\} \vee \{C_G(c) \wedge C_G(d)\}$
$= C_{C(G)}(x) \wedge C_{C(G)}(y)$.
If $C_{C(G)}(xy) \neq 0$, then $C(G)$ is a submultigroup of G.

Theorem 3.5 Let G be a multigroup over X, then $C(G) \trianglelefteq G$.
Proof Suppose $C(G) \sqsubseteq G$, then $C_{C(G)}(x) > 0, x \in X$ for any
$x \in X, C_{C(G)}(xy) = C_{C(G)}(xy), y \in (C(G))_*$
$C_{C(G)}(x^{-1}y^{-1}xy) \geq C_{C(G)}(x^{-1}) \wedge C_{C(G)}(y^{-1}xy)$
Since $\exists x(C(G))_* : C_{C(G)}(xx^{-1}) = C_{C(G)}(e)$
$\therefore C_{C(G)}(x) \wedge C_{C(G)}(x) = C_{C(G)}(x)$.
$\Longrightarrow C_{C(G)}(xyx^{-1}) \geq C_{C(G)}(y)$
Therefore, $C(G)$ is a normal submultigroup.

Theorem 3.6 Let $G \in MG(X)$, then $C(A) = G$ if G is abelian.

Proof Suppose $G \in MG(X)$ such that for any $x \in X$, $C_G(xy) = C_G(yx), y \in X$
$\Longrightarrow C_{C(G)}(xyx^{-1}) \geq C_{C(G)}(y), \forall y \in (C(G))_*$
$\Longrightarrow x \in X, C_{C(G)}(xyx^{-1}y^{-1}) = C_{C(G)}(e), \forall y \in (C(G))_*$
Then for all $x \in X$ we have $C_{C(G)}(xy) = C_G(yx), y \in X$
Hence, $C(G) = G$.

Theorem 3.7 Let $G \in MG(X)$ then center of X is $C(G))_*$
Proof Let $G \in MG(X)$ and $X = G_*$ such that $Z(X) = \{x \in X \mid xy = yx, \forall y \in X$, and for all $a \in Z(X)\ \exists\ a^{-1} \in Z(X)$ which implies that $C_{C(G)}(xy) = C_{C(G)}(x^{-1}y^{-1}), \forall x, y \in (C(G))_*$
Hence, $(C(G))_* = X$.

Theorem 3.8 Let G be a multigroup over X and $A, B \sqsubseteq G$ such that $C_A(e) = C_B(e), e \in X$ then $C(A) \cap C(B) = C(A \cap B)$.
Proof Suppose $x \in C(A \cap B) \Leftrightarrow C_{C(A\cap B)}(e) = C_{(A\cap B)}(xyx^{-1}y^{-1})$, for $x \in X$ and $\forall y \in X$

$$\Leftrightarrow C_A(e) = C_B(e) \Longrightarrow C_A(xyx^{-1}y^{-1}) = C_B(xyx^{-1}y^{-1})$$

$$\Leftrightarrow x \in C(A))_* \text{ and } x \in C(B))_*$$

Similarly, $C_{C(A)}(x) \neq 0$ and $C_{C(B)}(x) \neq 0 \Leftrightarrow C_{A\cap B}(x) \neq 0$

$$\Longrightarrow C_{C(A)}(x) \wedge C_{C(B)}(x) = C_{C(A\cap B)}(e)$$

Definition 3.9 Let $H \sqsubseteq G \in MG(X)$. The normalizer of submultigroup H of G, denoted by $N(G)$ is the multiset of G such that

$$N(G) := \begin{cases} C_G(x) \text{ if for any } x \in H_*, & C_G(xy) = C_G(yx), \forall y \in X \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.10 Let G be a multigroup over X, then

i. $N(H) \sqsubseteq G$,
ii. $H \trianglelefteq G$,
iii. $N(H)$ is normal submultigroup of G with the highest multiplicity.

Proof

i. Suppose $G \in MG(X)$ and $H \sqsubset G$ such that $\forall x \in (N(H))_*$ then $C_{N(H)}(x) = C_{N(H)}(x^{-1}) > 0$.
$C_{N(H)}(xy) = C_{N(H)}(yx), \forall y \in X$
For any $x \in (N(H))_*$ there exist $x^{-1} \in (N(H))_*$, $C_{N(H)}(xy)^{-1} = C_{N(H)}(yx)^{-1}$.

Cite this article as:
Michael, B.D., Nwagrabe, E., & Awolola, J.A. (2024). The commutative properties of multigroup. *FNAS Journal of Mathematical and Statistical Computing*, *2*(1), 39-43.

$$\Longrightarrow \forall x \in X, C_{N(H)}(x^{-1}y^{-1}) = C_{N(H)}(y^{-1}x^{-1}),$$
$$\Longrightarrow C_{N(H)}(yx) \geq C_{N(H)}(y) \wedge C_{N(H)}(x^{-1}).$$

Therefore, $C_G(x) \geq C_{N(H)}(x), \forall x \in X$. For any $x \in H_*$,

$$C_{N(G)}(x) = \vee C_G(x), C_G(xy) = C_G(yx), \forall y \in X.$$

ii. The result is obvious as Theorem 3.2

iii. Similar to Theorem 3.5

Theorem 3.11 Let G be an abelian multigroup over X, then $N(H) = G$.

Proof Let $G \in MG(X)$ such that $C_G(xy) = C_G(yx), \forall\, x, y \in X$. For any $H \sqsubseteq G$, we have
$\forall x \in X, C_{N(H)}(xy) = C_{N(H)}(yx), \forall y \in H_*$
Since $H \sqsubseteq G$, $C_{N(H)}(xy) = C_G(xy), \forall\, x, y \in X$
$\Leftrightarrow C_G(xy) = C_G(yx), \forall\, x, y \in X$.
Hence, $C_{N(H)}(x) =. C_G(x), \forall x \in X$.

Proposition 3.12 Given that G is a multigroup and $A, B \sqsubseteq G$, if $A \cap B$ then $N(A) \cap N(B) \sqsubseteq N(A \cap B)$

Definition 3.13 Centralizer of submultiset
Let $K \sqsubseteq G \in MG(X)$. The centralizer of an element $a \in G$, denoted by $Z(a)$ is a multiset of G generated by

$$Z(a) := \begin{cases} C_G(a) \text{ if for any } x \;\; C_G(ay) = C_G(ya), \forall y \in X \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Let $K \subseteq G \in MG(X)$. The centralizer of submultiset K of G, denoted by $Z(K)$ is the multiset of G defined by

$$Z(G) := \begin{cases} C_G(x) \text{ if for any } x \in K_*, \quad C_G(xy) = C_G(yx), \forall y \in X \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Clearly, $C_{Z(G)}(x) \neq 0$ since $\exists\, b \in \left(Z(G)\right)_* \mid C_G(by) = C_G(yb), \forall y \in X$ and for any
$x \in K_*, C_G(xy) = C_G(yx), \forall y \in X$.

Remark 3.14 Let $K \subseteq G \in MG(X)$, then the following holds;

i. $Z(G) \subseteq G$,

ii. $Z(K) = C_G(xyx^{-1}y^{-1}) = C_G(e), \forall x, y \in K_*$.

iii. $Z(H) = G$ if $C_G(xy) = C_G(yx), \forall x, y \in K_*$

Conclusion

The notion of commutative multigroup has been established with some results. Hence center of multigroup, centralizer of submultiset and normalizer of submultigroups is proposed in this paper alongside with some results to capture the multiplicity of the element to enhance further study of groups with multiplicity of elements

References

Awolola, J. A., & Ibrahim, A. M. (2016). Some results on multigroups. Quasi-groups and Related Systems, 24, 169-
Awolola, J. A., & Ibrahim, A. M. (2016). Some results on multigroups. Quasi-groups and Related Systems, 24, 169-177.

Ejegwa, P. A., & Ibrahim A. M., (2020). Some properties of multigroups, *Palestine Journal of Mathematics. 9*(1), 31 – 47.

Ejegwa, P. A. (2017). Upper and lower cuts of multigroups, *Prajna Int. J. Mathl. 1*(1), 19 – 26

Gyam, G. J., & Ogwola, P. (2023) Centralizer and normalizer of a multigroup, *International Journal of Recent Research in Mathematics Computer Science and Information Technology, 9*(2) 42 – 54.

Michael, B. D., & Ibrahim, M. A. (2023) On commutative submultigroups, Kristu Jayanti Journal of Computational Sciences (KJCS), 3(1), 23 - 29. https//doi.org/10.59176/kjcs.v3i1.2292
multisets, Novi Sad J. Math., 33(2), 73-92.

Nazmul S. K., Majumda P.,& Samanta . S. K (2013) On multisets and multigroups, Annals of Fuzzy mathematics and informatics, 6(3) 643 - 656

Nazmul, S. K., & Samanta, S. K. (2011). Fuzzy soft group, *Journal of Fuzzy Mathematics 19*(1), 101-114.

Rosenfeld, A. Fuzzy subgroups, (1971). *J. of Math. Appl*,.35, 512 – 517.

Shinoj, T. K., & Sunil, J. J. (2015). Intuitionistic fuzzy multigroups. Annals of Pure and Applied Mathematics, 9(1), 133 - 143.
Shinoj, T. K., Baby A., & Sunil, J. J. (2015). On some algebraic structures of fuzzy multisets. Annals of fuzzy Mathematics, 9(1), 77 - 90.
Singh, D., Ibrahim, A. M., Yohanna T., & Singh J. N., (2007). An overview of the application of