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Idempotents In Semigroups: Structure, Classification, Extension and Applications

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Abstract
This paper examines the structure, classification and extension of idempotents in semigroups, exploring their fundamental algebraic structures. Different identification and extension of idempotents in semigroups are presented. The various forms of examining idempotent behavior under different semigroup operations and their relationships with other elements are reviewed. Also, the distribution of idempotents within different classes of semigroups, such as regular semigroups, finite semigroups, and infinite semigroups are explained. The findings provides a deeper understanding of idempotent in semigroup structure, giving insights into the role of idempotents in more complex algebraic systems. Applications of idempotents in various fields are also presented.

Keywords: Classification, Extension of Idempotents, Idempotents, Semigroups, Structure

Introduction

Semigroup is a fundamental concept in modern algebra. In the realm of abstract algebra, semigroups were introduced independently by Wagner and Preston in Howie (2013) .The study of semigroup stands as a fundamental area, encapsulating structure with binary operation that adhere to the associative property. A semigroup is a set on which an associative operation is defined. The concept is a generalization of the structure of a group whereby half of the group axioms remains; hence the term "semigroup" (Abubakar, 2022).

A semigroup is an algebraic structure consisting of a set equipped with an associative binary operation (Szendrei & Zhang, 2016). Formally, a set S with a binary operation

$$\bullet \quad : \; S \times S \to S$$

is called a semigroup if it satisfies the following properties:

- Closure : $\forall\, a, b \in S,\; a * b \in S$
- Associativity : $\forall\, a, b, c \in S,\; a * (b * c) = (a * b) * c$

$(S, *)$ is called a semigroup .

An idempotent element in a set S is an element e ∈ S that satisfies the equation

$$e \cdot e = e$$

The term idempotent was first introduced by Pierce in the 19th century initially studied in the content of matrix algebra, endomorphism's idempotent elements were later explored within semigroup and other algebraic system. The study of idempotent has led to a deeper understanding of algebraic structure contributing to the development of modern algebra. The concept of idempotent has been a topic of interest since the early development of algebra (Bui-Xuan & Dejean, 2012). The study of idempotent elements in semigroups is significant because they often reveal important structural properties of the semigroup (Yamada, 2023). To identify idempotent elements in a semigroup S, one must find all elements e ∈ S such that $e * e = e$.

For finite semigroups , it's a systematical check of each element to see if it is an idempotent. Every non-empty finite semigroup has at least one idempotent. For infinite semigroups , algebraic methods or properties of the semigroup is used to narrow down potential idempotents (Almedia & Zeitoum, 2010).

A special type of infinitely long sentence which seems of fairly general applicability in the study of structure of bands (idempotent semigroups) is of importance. The technique is applied to a special type of band, namely normal bands which was used to obtain a rather complete structure and partial structural results on normal bands were obtained .Consider the semigroup (ℕ, min), where ℕ is the set of natural numbers and the operation is taking the minimum of two numbers. To find the idempotent elements: for any $n \in \mathbb{N}, \min(n,n) = n$ .Therefore, every element in $\mathbb{N}$ is idempotent under this operation.

Characterizing idempotents involves understanding their role and properties within the semigroup. In any semigroup, idempotents are precisely the fixed points of the function $f(x) = x\ .\ x$ .The study of idempotents within the framework of semigroups is an important and rich area of research in abstract algebra. Structures are essentials in algebra. In regular semigroup, idempotent plays an important role in the structure of semigroup .Several researchers have also studied the structure of regular semigroup based on certain identities satisfied by the set of idempotents.
Chrislock (1969) dealt with semigroup which satisfy the identity : $xaby = xbay$ called the medial semigroups and showed that regular medial semigroup are those semigroup which are orthodox normal bands of abelian groups. A generalized inverse semigroup is a regular semigroup with the idempotents satisfying a permutation identity thus

$$x_1 x_2 \ . \ . \ . x_n = x_{P_1}, x_{P_2}, x_{P_3}, \ . \ . \ . \ , x_{P_n} \text{ where } (P_1, P_2, \ . \ . \ . \ , P_n)$$

is a non trivial permutation of $(1,2, \ . \ . \ . \ , n)$. Blyth and Almeida (2008) introduced the notion of skew pair in the sense that an ordered pair (e, f) of idempotents of a regular semigroup is called a skew pair of $eF \in E$ where as $Fe \in E$. An idempotent element U of T is said to be quasi-normal idempotent, if $eUe = e$ for every $e \in E\ and\ uEu$ is a semilattice. Idempotents are seen to be immensely useful in the study of the structure of regular semigroups.

Despite the significance of idempotent in semigroup theory the structure and classification of idempotent within various classes of semigroup remains an area of ongoing research while certain classes of semigroup have well established results regarding idempotent elements. The identification and characterization of these idempotent elements hold significant theoretical and practical value for various reasons.

In many mathematical problems, idempotents serve as simplification tools. For instance, in the study of semigroup representations, knowing the idempotents can help break down complex problems into simpler, more manageable subproblems. This is because idempotent elements often correspond to important substructures, such as invariant subspaces in linear algebra. In ring theory: idempotents are essential in the decomposition of modules and rings into direct sums, while in semigroup theory, they help define regular semigroups and Green's relations (Barto & Kosik, 2014). The study of idempotents also extends to functional analysis, where they are used in the context of projections in Hilbert spaces. The structural properties of idempotents, such as their classification, behavior under various operations, and their role in the construction of orthogonal decompositions, provide insights into the underlying algebraic and geometric structure of mathematical objects and the classification of idempotents in more complex algebraic settings, exploring their applications in both pure and applied mathematics in the area.

The behavior of idempotents can be employed to explore various theoretical aspects, such as semigroup homomorphisms, subsemigroup structures, and the lattice of idempotents. This comparison helps in generalizing results and applying them to a much broader context. The comparative analysis of idempotents also illustrate how idempotents behave differently in any semigroup. This approach aids in understanding the nuances and exceptions in semigroup theory, links to other areas of algebra and mathematics, such as group theory and lattice theory, highlighting the interplay nature of semigroup studies.

By comparing idempotents across various semigroups, researchers can gain insights into the semigroup's complete structure. For example, the distribution and number of idempotents can reveal whether a semigroup is regular, completely regular, or has specific types of decomposition property. Different types of semigroups, such as commutative, regular, or inverse semigroups, have distinct properties related to their idempotents. A comparative analysis helps classify semigroups based on their idempotent structure, facilitating a deeper understanding of their classification and types (McFadden, & Ruskuc, 2015). Furthermore, applications of idempotents abound in semigroups and other fields (Kolokoltsov &Maslov,1997) as will be highlighted in this paper and more application are still being discovered.

This paper presents the structure ,classification of idempotents in specific classes of semigroups, presenting the conditions under which a semigroup has unique idempotent or a fixed number of idempotent and the extension and applications of idempotent in semigroup and their significance in algebraic structure .

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Structure Of Idempotents

i. Idempotent Identification: - Idempotent identification refers to the process of recognizing or determining elements within an algebraic structure (such as a semigroup, ring, or matrix) that have the idempotent property. An element is idempotent if, when operated on itself ,it produces the same element. Mathematically, an element e is idempotent if $e^2 = e$. Identifying idempotent elements is crucial in analyzing the structure's internal symmetries, decompositions, and stability, making it a fundamental concept in various areas of mathematics (Ren & Xu,2020).

ii. Matrix Representations:-In some cases, semigroups are represented using matrices (matrix semigroups), and computational methods involve matrix operations to analyze idempotent behavior. Techniques such as matrix multiplication, eigenvalue analysis, and matrix factorization can be employed for idempotent-related computations (Fleischer & Schwartz, 2013).

iii. Graph-Based Approaches:-Graph representations of semigroups are utilized for computational analysis of idempotents. Graph algorithms, such as depth-first search or topological sorting, can be adapted to explore idempotent-related properties and relationships (Abubakar & David, 2016).

iv Parallel and Distributed Computing:-For computational complexity of idempotent analysis in large semigroups, parallel and distributed computing techniques are applied. These methods leverage multiple processors or distributed computing resources to expedite idempotent-related computations (Howie, 2011).

v. Optimization and Heuristics:-Optimization techniques and heuristics are employed to improve the efficacy of idempotent-related algorithms. This includes strategies such as pruning search spaces, using heuristics to guide computations, optimizing data structures for faster access, algorithm for continuous -time stochastic control and discrete time stochastic control (McEneaney, 2009).

vi. Application-Specific Algorithms: Application-specific algorithms are algorithms designed and optimized for a particular task or problem within a specific domain or application. These algorithms are fashioned to leverage the unique characteristics and constraints of the application, making them more efficient and effective for that particular use case compared to general-purpose algorithms. For example, an application-specific algorithm for image compression in medical imaging would be designed to preserve critical details while minimizing file size, optimizing performance for that specific context. Computational methods are tailored to specific applications of idempotents, such as in automata theory, cryptography or optimization. Algorithms are designed to address the unique requirements and constraints of these application domains while carefully analyzing their idempotents properties

Understanding idempotents in semigroup Structure

Idempotents play a crucial role in understanding the internal structure of semigroups. By characterizing idempotent elements, mathematicians can gain insights into the broader structural properties of the semigroup. This is fundamental to advancing the theory of semigroups and related algebraic systems thus:

i.Subsemigroups: The set of all idempotents in a semigroup S, denoted as *E(S)*, forms a subsemigroup of S. This subsemigroup often has special properties and can provide insights into the structure of S.

ii. Commutativity: If e and f are idempotents

$$e * f = f * e$$

They also play a crucial role in the context of Green's relations, in understanding the structure of the semigroup. Green's relation are five equivalence relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathfrak{J}$ D,that characterize the elements of a semigroup in terms of the principal ideals they generate. The relations are named after James Alexander Green, who introduced them. John Mackintosh Howie, a prominent semigroup theorist, remarked that on encountering a new semigroup, almost the first question one asks is 'What are the Greens relations like?' For elements a and b of S, Green relations, $\mathscr{L}$, $\mathscr{R}$, $\mathfrak{J}$, $\mathscr{H}$, D, are defined by

$$a \mathscr{L} b \quad iff \quad Sa = Sb$$
$$a \mathscr{R} b \quad iff \quad aS = bS$$
$$a \mathfrak{J} b \quad iff \quad SaS = SbS$$
$$a \mathscr{H} b \quad iff \quad a \mathscr{L} b \text{ and}$$
$$a \mathscr{R} b \; a D b$$

If and only if there exists a c in S such that $a \mathscr{L} c$ and $c \mathscr{R} b$ That is, a and b are $\mathscr{L}-$related if they generate the same left ideal , $\mathscr{R}$-related if they generate the same right ideal and $\mathfrak{J}$ −related if they generate the same two-sided ideal. These are equivalence relations on S, so each of them yields a partition of S (Howie, 2005). For instance, H-classes containing idempotents and are particularly significant (Lallement , 2018).

iii. On the concept of minimality in a finite semigroup, every idempotent e is contained in a minimal ideal, which is often a simple semigroup (Bulman-Fleming & Jones, 2013).

iv. For equivalence classes: Idempotents often partition the semigroup into equivalence classes, where each class is associated with a particular idempotent (Costa & Nogueira, 2017).

Like in the semigroup (M_n, ∘) of $n \times n$ matrices over a field under matrix multiplication, an idempotent matrix E satisfies $E^2 = E$. These matrices are projections, meaning they project the vector space onto a subspace.
v. Structural Insights: Idempotents help reveal the underlying structure of semigroups, particularly in regular semigroups where they form key components analysis of the structure
vi. Simplification of complex structures: By classifying idempotents, mathematicians can simplify the study of complex semigroups by breaking them down into more manageable components where each is associated with a specific idempotent class.
vii. Projection Properties: Idempotents in semigroups can be thought of as projection elements, with ea "projecting" a onto the "space" determined by e. This projection property is essential in various applications, such as in linear algebra and optimization problems (Costa, 2017).
viii. Idempotents as zero divisors**:** In semigroups with zero divisors and elements a and b where

$$ab = 0 \text{ or } ba = 0,$$

idempotents are often related to the presence or absence of zero divisors. In a semigroup without zero divisors, idempotents can be used to characterize their structural properties.
ix. Inverses: Idempotents are closely related to the existence of inverses in semigroups. In particular, idempotent elements may act as self-inverses (i.e., $ee = e$ implies e is its own inverse.
x. Semilattice structure :- In semigroups that are also semilattices, idempotents play a central role (Lawson, 2004).
A semilattice is a structure where every pair of elements has a least upper bound (join) and a greatest lower bound (meet). Idempotents are precisely the elements that are equal to their join and meet.
xi. Regularity:- In Regular semigroups S, which satisfies the property

$$axa = a \;\; \forall\, a \in S$$

have a strong relationship with idempotents. In regular semigroups, idempotents serve as both left and right identities for elements, contributing to the regularity property. By exploring these structural properties and relationships, researchers gain insights into the intrinsic nature of idempotents within semigroups, uncovering their rich algebraic structure and applications in various mathematical contexts (Dolinka, 2015).

Classification Of Idempotents In Semigroups

The classification of idempotents in semigroups is a central topic in semigroup theory, focusing on understanding how these elements organize within a semigroup. The study of these elements helps in classifying semigroups, especially regular semigroups, where idempotents play a critical role in defining Green's relations that describe the structure and decomposition of semigroups (Gomes & Gould, 2011).

Classifying idempotents involves identifying and organizing them into specific classes, often using algebraic tools such as Green's relations, and understanding their interactions within the semigroup. This classification not only sheds light on the internal symmetry and structure of the semigroup but also provides a pathway to understanding the semigroup's behavior under various operations, including homomorphisms and products (Gomes & Gould, 2011).

Studying the classification of idempotents is essential for:

i.Idempotence (Idempotent closure): - Idempotents form a closed set within a semigroup, meaning that if e and f are idempotents, then ef is also an idempotent implying that the operation can be applied multiple times without a change in the initial application .
ii.Idempotent-generated subsemigroups: - The set of all idempotents in a semigroup generates a subsemigroup called the idempotent-generated subsemigroup. This subsemigroup often exhibits interesting properties and relationships with the original semigroup (Auinger & Steinberg, 2011).
iii.Central idempotents: Central idempotents are idempotent elements that commute with all elements of the semigroup. They play a crucial role in characterizing the centre of a semigroup and understanding its structure.
ivIdempotent-Generated Sub semigroups: - Idempotent-generated subsemigroups are subsemigroups within a semigroup that are generated by the semigroup's idempotent elements. In other words, these subsemigroups are formed by taking all possible combinations and products of the idempotent elements within the semigroup. These sub semigroups are important because they often capture key structural properties of the semigroup, such as regularity and decomposition patterns. Studying idempotent-generated sub semigroups helps in understanding the overall behavior and classification of the semigroup (Harju & Peter, 2012).
Let S be a semigroup, and let $E(S)$ denote the set of all idempotent elements in S, i.e.,

$$E(S) = \{e \in S \;/\; e^2 = e\}$$

The idempotent-generated sub semigroup of S is the sub semigroup generated by the set $E(S)$. Formally, it is defined as:

$$\langle E(S)\rangle = \{x_1, x_2, \;.\;.\;.\;, x_n \;/\; x_i \in E(S), \forall\, i \in \{1,2, \;.\;.\;.\;, n\}, n \in N\}$$

The cyclic set $\langle E(S)\rangle$ consists of all finite products of idempotent elements from S and is itself a subsemigroup of S. Computational methods are used to generate the idempotent-generated subsemigroup of a given semigroup.

This involves determining all idempotents in the semigroup and computing the closure under the specific semigroup operation.
v. Weakly idempotent elements:-Weakly idempotent elements are generalizations of idempotents that satisfy a weaker form of the idempotent condition. Instead of , weakly idempotent elements satisfy for some idempotent element .These elements have applications in contexts where strict idempotence is not required but a form of self-preservation is much desirable.
vi. Partial idempotents:-Partial idempotents are idempotent elements defined on subsets or partial operations within a semigroup. They capture idempotent behavior within restricted domains or operations, providing insights into localized idempotent structures, (Dolinka, 2015).In programming , a partial idempotent latin square is a partial latin square with the property that all the cells on the main diagonal are occupied and furthermore cell (i, i) is occupied by i (Linder, 1991).
vii. Idempotent semirings and rings:-Idempotent semirings and rings are algebraic structures that combine either semigroup or ring operations with idempotent properties. They have applications in algebraic coding theory, optimization problems, and mathematical modeling.
viii. Idempotent-generated structures:-Generalizations involve studying structures generated by idempotent elements, such as idempotent-generated subsemigroups, rings or lattices. These structures clearly exhibit unique properties and relationships due to the idempotent generation process (Costa & Nogueira, 2017; Gan &Zhang, 2014)
ix .Idempotent modules and algebras:-Extensions to module theory and algebraic structures involve incorporating idempotent elements into module and algebraic operations. Idempotent modules and algebras provide frameworks for studying idempotent-related interactions (Carlson, 1996).
x. Non-Associative and non-commutative settings:-Generalizations of idempotents to non-associative and non-commutative settings explore idempotent-like elements in diverse algebraic structures. These settings may involve non-standard operations or alternative algebraic properties.
xi. Dualities and complementary structures:-Dualities and complementary structures explore relationships between idempotent and non-idempotent elements within a semigroups. This includes studying complementary pairs of idempotents or duality relationships that arise in specific semigroup classes.
xii. Topological idempotents :-Generalizations of idempotents in topological semigroups and topological spaces involve studying idempotent-related properties in continuous and topological settings. Topological idempotents and their structural properties contribute to understanding topological semigroups and related algebraic structures.
In topology, an idempotent of a space Y is a map $\xi : Y \to Y$ such that $e^2 = e$ where there is a one-one correspondence between retracts of Y and idempotents of Y (Arkowitz, 1995). A comparative analysis of idempotents in semigroups involves examining and contrasting the properties, structures, and roles of idempotent elements within different semigroups. Idempotents play a pivotal role in understanding the underlying algebraic structure of semigroups (Lawson, 2021).
xiii.Idempotent matrices (Matrix semigroup) **:** An idempotent matrix is a matrix *A* such that when it is multiplied by itself, the result is the matrix itself. In other words, a matrix *A* is idempotent if $A^2 = A$. This property implies that the matrix does not change when squared.
xiv. Direct Products:- Idempotents in direct products of semigroups can be classified based on their components in their direct product structure. Understanding how idempotents interact across product components is essential for their classification in direct product semigroups (Yamada, 2023).

Generalizations And Extensions Of Idempotents In Semigroups

i. Regular Semigroups: In a regular semigroup, every element has a unique idempotent as it's right and left identity. Classification involves identifying principal idempotents alongside those generating regular subsemigroups and non-principal idempotents. The Green's relations (like Green's J-class) play a crucial role in classifying idempotents in regular semigroups (Higgins, 2015).Thomas et al., (2018) reviewed several idempotent extensions of regular semigroups :

> For a regular semigroup T, the elements $eaa, a'ea$ and e are idempotents for all $a \in T$, $e \in E, a' \in V(a)$ and $e' \in V(e)$. Set of idempotents of T gives a normal band. They reiterated that some conditions can be given on idempotent of regular semigroups to obtain idempotents such as medial, weak medial, normal, quasi-normal, middle unit, weak middle unit , one sided identity and identity.

ii. Commutative semigroups:- In commutative semigroups, idempotents commute with all elements. Classification involves understanding the structure of idempotent-generated subsemigroups, which can be monogenic (generated by a single idempotent) or non-monogenic.

iii. Bands: -In Bands, every element is idempotent. Classification focuses on identifying primitive idempotents (those that cannot be expressed as a sum of two non-zero idempotents) and non-primitive idempotents (Howie &Marques-Smith, 2011).

iv. Cancellative semigroups:- Cancellative semigroups have the property that

$$ab = ac \ \Rightarrow\ a = c \text{ and } ba = ca \ \Rightarrow\ b = c$$

Classification involves characterizing idempotent elements in cancellative semigroups based on their interaction with cancellative properties.

v. Simple semigroups: 0-simple semigroups have a unique minimal ideal. Classification primarily is about the idempotents related to this minimal ideal, such as non-zero idempotents and zero idempotents (Howie, 2013).

vi. Maximal subgroups: In semigroups with maximal subgroups, idempotents often play a crucial role in understanding the structure and properties of these subgroups. Classification involves studying idempotents that are central to the maximal subgroup structure and their relationships with other elements.

vii. Specialized classes**:** Special classes of semigroups, such as orthodox semigroups, completely 0-simple semigroups, and regular band semigroups, have specific idempotent classification criteria.

Applications Of Idempotents In Semigroups

i. Automata theory and formal languages:-Idempotents are critical in automata theory and the study of formal languages. Semigroups and their idempotent elements can be used to model state transitions in automata, where idempotents often represent stable states. Understanding these elements can lead to more efficient algorithms for automata minimization and language recognition (Pin, 2012).

ii. Idempotent algebras:- Idempotent algebras can be obtained using specific operations defined on the idempotents of a semigroup.

iii. Category theory and universal algebra :-The results from the study of idempotents in semigroups contribute to broader fields such as universal algebra and category theory. For example, in category theory, idempotent elements correspond to idempotent endomorphisms, which are important in the study of monads and adjunctions.

iv. Enhancement of computational algebra systems:-The efficient computation and identification of idempotents in computational algebra are vital for developing software that handles algebraic structures. Improved algorithms for identifying and working with idempotents enhances capabilities of computer algebra systems which benefits both theoretical research and practical applications

v. Graph generating functions:- The finiteness of an idempotents in a semigroup can be ascertained using graph theoretic functions computational techniques for idempotents in semigroups

vi. Computational methods for classifying idempotents often rely on algorithms that explore semigroup structure, Green's relations and properties specific to the semigroup class. Efficient algorithms for identifying principal idempotents, computing idempotent-generated subsemigroups and analyzing idempotent-related properties contribute to classification charts. The structural properties and relationships of idempotents in semigroups are crucial for understanding their behavior and role within these algebraic structures as reported for Restriction semigroups (Higgins & Gould, 2011; Abubakar, 2022).

vii. Semigroups generation description : A bijection in a finite semigroup like Transformation semigroup is a criteria for an element to be expressed as a product of idempotents .

Conclusion

Idempotents play a significant role in semigroups through its structure, function’s behaviour, operations defined or imposed on them bringing out semigroups which can lead to bigger structures, redefining concepts , giving insights into mode of semigroups etc. With all that has been highlighted in this paper, idempotents can be looked as a significant concept in semigroups.

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