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The Embedding of Category of Type A Monoid into Inverse Semigroup Categories with their Translational Hulls

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Abstract
According to Fountain (1979), type A semigroup is characterized as follows: that S is a type A semigroup if and only if there are inverse semigroups S_1, S_2, and embeddings $\phi_1 : S \to S_1$, $\phi_2 : S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_1 a)(\phi_1 a)^{-1}$. With full transformation semigroup, this characterization leads to faithful representation of type A semigroup. Offor et al. (2018) extended the representation to the translational hull of type A semigroup. In this paper, we are further extending the representation to the category of the translational hull of type A monoid.

Keywords: Translational Hull, Category, Embedding, Binary Operations, Mapping

Introduction

Let S be a set and $\theta : S \times S \to S$ a binary operation that maps each ordered pair (x, y) of S to an element $\theta(x, y)$ of S. The pair (S, θ) (or just S, if there is no fear of ambiguity) is called a groupoid. The mapping θ is called a product of (S, θ). $x \cdot y$, xy and $\theta(x, y)$ all mean the same and are called the product of x and y.
A groupoid S is called a semigroup, if the operation θ is associative. That is, for all $x, y, z \in S$, $\theta(x, \theta(y, z)) = \theta(\theta(x, y), z)$. A semigroup is a *monoid*, if it has an identity. An element e in a semigroup such that $e^2 = e$ is called an *idempotent*. We denote the set of idempotents of a semigroup S by E_S.

Let $(S_1, \cdot)$ and $(S_2, *)$ be two semigroups. A mapping $\alpha : S_1 \to S_2$ is a *homomorphism*, if $\forall x, y \in S$, $\alpha(x \cdot y) = \alpha(x) * \alpha(y)$. It is an *embedding* if $\alpha(x) = \alpha(y)$ implies $x = y$ and if in addition, $\forall y \in S_2$, $\exists x \in S_1$ with $\alpha(x) = y$, then α is called an isomorphism. The *kernel* as the relation $ker(\alpha) = \{(x, y) \mid \alpha(x) = \alpha(y)\}$.
Let X be a set, and denote by T_X the set of all functions $\alpha : X \to X$. T_X is called the *full transformation* semigroup on X with the operation of composition of functions. A homomorphism $\phi : S \to T_X$ is called the *representation* of the semigroup S. The set of all partial one-one maps of any non-empty set X is an inverse semigroup and it is called symmetric inverse semigroup usually denoted by $\mathfrak{T}_X$.
Let S be a semigroup and $a, b \in S$. $(a, b) \in \mathcal{L}^*$ if $\forall x, y \in S^1$, $ax = ay$ if and if $bx = by$. $\mathcal{R}^*$ is dual to $\mathcal{L}^*$ and this definition of $\mathcal{L}^*$ apply in dual manner to $\mathcal{R}^*$. The intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. Reader should read up the Green's Equivalences and the *-Equivalences. See Lawson (1986), Asibong-Ibe (1991), Howie (1995) , and Offor et al. (2018).

Some Basic Semigroup Theories

The Natural Order Relation on an Inverse Semigroup

According to Lawson (1987), it is possible to define a partial order on an inverse semigroup as follows:

Given a, b in an inverse semigroup S with semilattice of idempotents E, $a \leq b$ if and only if $\exists e \in E$ such that $a = eb$. We show here that the relation is a partial order.
Since $a = (aa^{-1})a$, $a \leq a$ and hence the relation is reflexive. For antisymmetry, let $a \leq b$ and $b \leq a$. Then $\exists e, f \in E$ such that $a = eb$ and $b = fa$ and it follows that

$$a = eb = efa = fea = feeb = feb = fa = b.$$

And for transitivity, assuming $a \leq b$ and $b \leq c$, then $\exists e, f \in E$ such that $a = eb$ and $b = fc$. It follows that $a = (ef)c$ and since $ef \in E$, $a \leq c$.
Furthermore, the order relation is compatible with the multiplication of S and to see this, we have to show that $[\, a \leq b$ and $c \in S] \Rightarrow [ac \leq bc$ and $ca \leq cb]$.
The first implication is straightforward since $a = eb$ implies that $ac = e(bc)$. And for the second implication, notice that if $a = eb$, then

$$ca = ceb = c[(c^{-1}c)e]b = c(ec^{-1}c)b = (cec^{-1})cb.$$

Finally, the relation is also compatible with inversion in the sense that $a \leq b \Rightarrow a^{-1} \leq b^{-1}$;
for $a = eb$ implies that $a^{-1} = b^{-1}e = b^{-1}bb^{-1}e = b^{-1}ebb^{-1} = (b^{-1}eb)b^{-1}$.

The Inverse Semigroup and The Type *A* Semigroup

A semigroup S is called an *inverse semigroup* if for all $a, b \in S$, $(a^{-1})^{-1} = a$, $aa^{-1}a = a$ and $aa^{-1}bb^{-1} = bb^{-1}aa^{-1}$
A semigroup is called $left(right)$ $abundant$ if each $\mathcal{R}^*$- ($\mathcal{L}^*$-) class contains an idempotent and $abundant$ if it is both left and right abundant. If the idempotents of a left (right) abundant semigroup form a semilattice, it is called $left(right)$ $adequate$. It is called $adequate$ if it is both left and right adequate.
In an adequate semigroup, the idempotents in each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class are unique. If S is adequate, and a is an element of S, then $a^*(a^\dagger)$ will denote the unique idempotent in the $\mathcal{L}^*$- ($\mathcal{R}^*$-)class of a.
A left(right) adequate semigroup S is called $left(right)$ $type\ A$ if $ae = (ae)^\dagger a[ea = a(ea)^*]$ for all $a \in S$ and all idempotents $e \in S$. An adequate semigroup is called $type\ A$ if it is both left and right type$A$.
Fountain (1979) characterised a type A semigroup as follows:

Lemma 2.2.1 Fountain (1979)**:** Let S be an adequate semigroup. Then, $\forall a \in S$ and $\forall e \in E(S)$, S is a type A semigroup if and only if $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$.

If S is an adequate semigroup with semilattice E of idempotents, then $\forall a, b \in S$, if $a\mathcal{L}^*b$ then $\mathcal{L}_a^* = \mathcal{L}_b^*$ and a^* is the unique idempotent in $\mathcal{L}_a^*$, b^* unique idempotent in $\mathcal{L}_b^*$. Therefore, $a^* = b^*$.
Conversely, if $a^* = b^*$ then $a^*\mathcal{L}^*b^*$ and we have $a\mathcal{L}^*a^*\mathcal{L}^*b^*\mathcal{L}^*b$. So that $a\mathcal{L}^*b$. Hence, $a\mathcal{L}^*b$ if and only if $a^* = b^*$ $(\forall a, b \in S)$. Dually, $(\forall a, b \in S)$ $a\mathcal{R}^*b$ if and only if $a^\dagger = b^\dagger$.
$a\mathcal{L}^*a^* \Rightarrow ab\mathcal{L}^*a^*b$ and therefore $(ab)^* = (a^*b)^*$. $b\mathcal{R}^*b^\dagger \Rightarrow ab\mathcal{R}^*ab^\dagger$ and therefore $(ab)^\dagger = (ab^\dagger)^\dagger$.
It is therefore obvious that for $e \in E$, $(ae)^* = a^*e$ and $(ea)^\dagger = ea^\dagger$. $(ab)^*$ and b^* are idempotents. Therefore, $(ab)^*b^* = [(ab)^*b^*]^* = (abb^*)^* = (ab)^*$. Thus, $b^*(ab)^* = (ab)^*b^* = (ab)^*$.
Therefore, $(ab)^* \leq b^*$, where $\leq$ is the usual ordering on E
Similarly, $(ab)^\dagger a^\dagger = a^\dagger(ab)^\dagger = [a^\dagger(ab)^\dagger]^\dagger = (a^\dagger ab)^\dagger = (ab)^\dagger$. Therefore, $(ab)^\dagger \leq a^\dagger$.

Theorem 2.2.2 Fountain (1979)**:** Let S be an adequate semigroup, then the following conditions are equivalent:

i. S is a type A semigroup
ii. $\forall a \in S$ and $\forall e \in E(S)$, $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$.
iii. there are inverse semigroups S_1, S_2, and embeddings $\phi_1 : S \to S_1$, $\phi_2 : S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_1 a)(\phi_1 a)^{-1}$.

Inverse Semigroup as a Member of Type *A* Semigroup

We know that an inverse semigroup S is regular with commuting idempotents. Therefore $\mathcal{R} = \mathcal{R}^*$, $\mathcal{L} = \mathcal{L}^*$ and thus, every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class contains a unique idempotent. Hence, S is adequate with $a^\dagger = aa^{-1}$ and $a^* = a^{-1}a$ which are respectively the unique idempotents in $\mathcal{R}^*$ and $\mathcal{L}^*$.
Now, $ea = e(aa^{-1})a = aa^{-1}(ea) = a(a^{-1}ea) = a(ea)^{-1}(ea) = a(ae)^*$

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and $ae = a(a^{-1}a)e = ae(a^{-1}a) = (aea^{-1})a = (ae)(ae)^{-1}a = (ae)^{\dagger}a$. Hence, S is type A.
An inverse semigroup is therefore a member of type A semigroup. It is therefore natural that some of the results in inverse semigroup can be generalized in type A.

THE TRANSLATIONAL HULL OF A SEMIGROUP
Preliminaries
According to Reilly (1974) a map λ from a semigroup S to itself is a $left\ translation$ of S if for all elements $a, b \in S$, $\lambda(ab) = (\lambda a)b$. A map ρ from a semigroup S to itself is a $right\ translation$ of S if $(ab)\rho = a(b\rho)$ for all elements $a, b \in S$. A left translation λ and a right translation ρ are linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$. The set of all linked pairs (λ, ρ) of left and right translations is called the $translational\ hull$ of S and it is denoted by $\Omega(S)$. We denote the set of all the idempotents of $\Omega(S)$ by $E_{\Omega(S)}$. The restriction of a function φ to a subset A of its domain is denoted by $\varphi|A$. The set of the left translations of S is denoted by $\Lambda(S)$ and the set of the right translations of S is denoted by $P(S)$. $\Omega(S)$ is a subsemigroup of the direct product $\Lambda(S) \times P(S)$. For $(\lambda, \rho)(\lambda', \rho') \in \Omega(S)$, the multiplication is given by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$ where $\lambda\lambda'$ denotes the composition of the left maps λ and λ' (that is, first λ' and then λ) and $\rho\rho'$ denotes the composition of the right maps ρ and ρ' (that is, first ρ and then ρ'). For each a in S, there is a linked pair (λ_a, ρ_a) within $\Omega(S)$ defined by $\lambda_a x = ax$ and $x\rho_a = xa$, and called the $inner\ part$ of $\Omega(S)$ and for all $a, b \in S$, $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$ is obvious and this gives another obvious fact that $a \mapsto (\lambda_a, \rho_a)$, a map from S into $\Omega(S)$, is a morphism. Petrich (1970) termed it the $canonical$ homomorphism of S into $\Omega(S)$. The left translation λ is symmetrical to the right translation ρ and therefore, properties of λ can be attributed symmetrically to ρ.

The Translational Hull of an Inverse Semigroup

The preliminaries above have given us some of the definitions and denotations we need in this section. One of the central information captured in that section is that the set of all linked pairs (λ, ρ) of left and right translations of a semigroup S is called the $translational\ hull$ of S and it is denoted by $\Omega(S)$. And we also noted the following:

$\Lambda(S) = \{\lambda: S \to S \mid \lambda(xy) = (\lambda x)y,\ \forall x, y \in S\}$

$P(S) = \{\rho: S \to S \mid (xy)\rho = x(y\rho),\ \forall x, y \in S\}$

The sets - $\Lambda(S)$ and $P(S)$ - are semigroups under composition of maps.

$\Omega(S) = \{(\lambda, \rho) \in \Lambda(S) \times P(S) |\ x(\lambda y) = (x\rho)y,\ \forall x, y \in S\}$

With multiplication defined by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$, $\Omega(S)$ is a semigroup.
We also made mention of $canonical$ homomorphism of S into $\Omega(S)$. Now, let us call this map Π_S and then for $a \in S$, we have $\Pi_S : a \mapsto (\lambda_a, \rho_a)$. $(\lambda_a, \rho_a) \in \Omega(S)$ and the set $\Pi_S(S) = \{(\lambda_a, \rho_a) |\ a \in S, \lambda_a x = ax,\ x\rho_a = xa,\ \forall x \in S\}$ is called the $inner\ part$ of $\Omega(S)$. The set idempotents of $\Lambda(S)$ is denoted $\Lambda(E)$ where E is the semilattice of idempotents S. Let $\Gamma(S) = \{\lambda_a: a \in S\}$.
Assuming $\lambda_a = \lambda_b$ and $\rho_a = \rho_b$. Let a' and b' be inverses of a and b respectively. Then $a = aa'a = (\lambda_a a')a = (\lambda_b a')a = ba'a$. $ba'aS \subseteq bS$ and therefore $aS \subseteq bS$ and thus $R_a \leq R_b$.
Similar arguments show that $R_b \leq R_a$, $L_a \leq L_b$, $L_b \leq L_a$ and so aHb and therefore $aa' = bb'$ and $a'a = b'b$. It then easily follows that $a = ba'a = bb'b = b$. Therefore, for an inverse semigroup S with $a, b \in S$, $[\lambda_a = \lambda_b\ ,\ \rho_a = \rho_b] \Rightarrow a = b$.
In any inverse semigroup S, $\lambda e = \lambda ee = (\lambda e)e = e(\lambda e) = (e\rho)e = e(e\rho) = e\rho \quad [e \in E_S\ ,\ (\lambda\ ,\rho) \in E_{\Omega(S)}]$

Theorem 3.2.1 Ault (1972): Translational hull of an inverse semigroup is an inverse semigroup.
Definition 3.2.2 Ault (1972): For $(\lambda, \rho) \in \Omega(S)$, the inverse $(\lambda, \rho)^{-1}$ is denoted by $(\lambda^{-1}, \rho^{-1})$ and is defined by

$\lambda^{-1}x = (x^{-1}\rho)^{-1}$, and $x\rho^{-1} = (\lambda x^{-1})^{-1} \quad \forall\, x \in S$.

The left translation λ is symmetrical to the right translation ρ and therefore, properties of λ is attributed symmetrically to ρ.
Let S be an inverse semigroup with the semilattice of idempotents E_S and $(\lambda, \rho) \in \Omega(S)$. Assuming $\lambda e = \lambda' e$ for all $e \in E_S$, and that a is an element of S. Then, $\lambda a = \lambda aa^{-1}a = \lambda(aa^{-1})a = \lambda'(aa^{-1})a = \lambda' aa^{-1}a = \lambda' a$. Thus, $\lambda = \lambda'$. The converse is obvious. That of ρ follows symmetrically. Thus, two left (right) translations of an inverse semigroup S are equal if they agree on all idempotents of S. That is, if $\lambda|_{E_S} = \lambda'|_{E_S}$ $(\rho|_{E_S} = \rho'|_{E_S})$.

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Furthermore, assuming (λ, ρ) be an idempotent in $\Omega(S)$. Then $\lambda^{-1} = \lambda,\ \rho^{-1} = \rho$ and therefore $e\rho = e\rho^{-1}$ for all $e \in E_S$. We know that $e\rho^{-1} = (\lambda e^{-1})^{-1} = (\lambda e)^{-1}$. Therefore, $e\rho = (\lambda e)^{-1}$. So that $\lambda e = (\lambda e)(\lambda e)^{-1}(\lambda e) = (\lambda e)(e\rho)(\lambda e) = (\lambda e e\rho)(\lambda e) = (\lambda e\rho)\lambda e = \lambda e(\lambda\lambda e) = (\lambda e)(\lambda e)$. Thus, $\lambda e \in E_S$. Similarly, $e\rho \in E_S$.
Conversely, let $\lambda(E_S) \subseteq E_S$. Then, $\lambda^2 e = \lambda(\lambda e) = \lambda(e(\lambda e)) = (\lambda e)(\lambda e) = \lambda e$. Therefore $\lambda^2 = \lambda$. By symmetry, we also have $\rho^2 = \rho$. Hence, $(\lambda, \rho)^2 = (\lambda, \rho)$. This shows that (λ, ρ) is idempotent if and only if $\lambda(E_S) \subseteq E_S$ and $(E_S)\rho \subseteq E_S$.

The Translational Hull of a Type A Semigroup

Let's start by noting that two left (right) translations of a type A semigroup are equal if they agree on all idempotents.
If a be an element of S and e be an idempotent in the $\mathcal{R}^*$-class of a, then $ea = a$ and so $\lambda a = \lambda(ea) = (\lambda e)a = (\lambda' e)a = \lambda'(ea) = \lambda' a$ and thus, $\lambda = \lambda'$. Thus, if the restrictions of $\lambda,\ \lambda'\ (\rho, \rho')$ to the set of idempotents of S are equal, $\lambda = \lambda'\ (\rho = \rho')$.

Assuming S is an adequate semigroup with semilattice of idempotents $E(S)$. Let $(\lambda, \rho) \in \Omega(S)$ and for all $a \in S$, define maps $\lambda^\dagger, \lambda^*, \rho^\dagger, \rho^*$ of S to itself as follows: $\lambda^\dagger a = (a^\dagger\rho)^\dagger a$; $\lambda^* a = (\lambda a^\dagger)^* a$
$a\rho^\dagger = a(a^*\rho)^\dagger$; $a\rho^* = a(\lambda a^*)^*$
And for $e \in E$, $\lambda^\dagger e = (\lambda e)^\dagger$; $\lambda^* e = (\lambda e)^*$; $e\rho^\dagger = (e\rho)^\dagger$; $e\rho^* = (e\rho)^*$
We notice from the definition that $\lambda^\dagger e,\ \lambda^* e,\ e\rho^\dagger$ and $e\rho^*$ are idempotents. We also need to note that $\lambda^\dagger b^\dagger$ and $a^*\rho^\dagger$ are idempotent of S since $\lambda^\dagger b^\dagger \,.\, \lambda^\dagger b^\dagger = \lambda^\dagger b^\dagger b^\dagger \rho^\dagger = \lambda^\dagger b^\dagger \rho^\dagger = \lambda^\dagger \lambda^\dagger b^\dagger = \lambda^\dagger b^\dagger$

And $a^*\rho^\dagger \,.\, a^*\rho^\dagger = \lambda^\dagger a^* \,.\, a^*\rho^\dagger = \lambda^\dagger a^*\rho^\dagger = a^*\rho^\dagger\rho^\dagger = a^*\rho^\dagger$

We note the following also:
i) $\lambda^\dagger e = (e\rho)^\dagger e = e(e\rho)^\dagger = e(e^*\rho)^\dagger = e\rho^\dagger$ ii) $\lambda^* e = (\lambda e)^* e = e(\lambda e)^* = e\rho^*$
Let $e \in E$. $(\lambda^*)^2 e = \lambda^*(\lambda^* e) = \lambda^*(\lambda e^\dagger)^* e = \lambda^* e(\lambda e)^* = (\lambda^* e)(\lambda^* e) = \lambda^* e$. So that $(\lambda^*)^2 = \lambda^*$
and $(\lambda^\dagger)^2 e = \lambda^\dagger(\lambda^\dagger e) = \lambda^\dagger(e^\dagger\rho)^\dagger e = \lambda^\dagger e(e^\dagger\rho)^\dagger = \lambda^\dagger e(\lambda e^\dagger)^\dagger = \lambda^\dagger e(\lambda e)^\dagger = (\lambda^\dagger e)(\lambda^\dagger e) = \lambda^\dagger e$
So that $(\lambda^\dagger)^2 = \lambda^\dagger$. In similar argument, we have $(\rho^*)^2 = \rho^*$ and $(\rho^\dagger)^2 = \rho^\dagger$.
Thus, for any member (λ, ρ) of $\Omega(S)$, the elements (λ^*, ρ^*) and $(\lambda^\dagger, \rho^\dagger)$ are idempotents.
Now, $\lambda^\dagger(ab) = [(ab)^\dagger\rho]^\dagger ab = [[(ab)^\dagger a^\dagger]\rho]^\dagger ab = [(ab)^\dagger(a^\dagger\rho)]^\dagger ab = (ab)^\dagger(a^\dagger\rho)^\dagger ab$
$= (a^\dagger\rho)^\dagger(ab)^\dagger ab = (a^\dagger\rho)^\dagger ab = (\lambda^\dagger a)b$. This shows that $\lambda^\dagger$ is a left translation.
Similarly, $(ab)\rho^* = ab[\lambda(ab)^*]^* = ab[\lambda(b^*)(ab)^*]^* = ab[(\lambda b^*)(ab)^*]^* = ab(\lambda b^*)^*(ab)^*$
$= ab(ab)^*(\lambda b^*)^* = ab(\lambda b^*)^* = a(b\rho^*)$. This shows that ρ^* is a right translation.
In the same vein, $\lambda^*(ab) = [\lambda(ab)^\dagger]^* ab = [\lambda[a^\dagger(ab)^\dagger]]^* ab = [(\lambda a^\dagger)(ab)^\dagger]^* ab = (\lambda a^\dagger)^*(ab)^\dagger ab$
$= (\lambda a^\dagger)^* ab = (\lambda^* a)b$. This shows that λ^* is a left translation
And, $(ab)\rho^\dagger = ab[(ab)^*\rho]^\dagger = ab[[(ab)^* b^*]\rho]^\dagger = ab[(ab)^*(b^*\rho)]^\dagger = ab(ab)^*(b^*\rho)^\dagger = ab(b^*\rho)^\dagger$
$= a(b\rho^\dagger)$. This shows that $\rho^\dagger$ is a right translation.
Furthermore, $a^*(\lambda^\dagger b^\dagger) = (\lambda^\dagger b^\dagger)a^* = \lambda^\dagger(b^\dagger a^*) = (b^\dagger a^*)\rho^\dagger = b^\dagger(a^*\rho^\dagger) = (a^*\rho^\dagger)b^\dagger$.
So that, $a(\lambda^\dagger b) = aa^*[\lambda^\dagger(b^\dagger b)] = aa^*\lambda^\dagger(b^\dagger b) = a\lambda^\dagger a^*(b^\dagger b) = a(a^*\rho^\dagger)b^\dagger b = (aa^*)\rho^\dagger(b^\dagger b) = (a\rho^\dagger)b$. This implies that $\lambda^\dagger$ and $\rho^\dagger$ are linked.

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And $a(\lambda^* b) = a(\lambda b^\dagger)^* b = a\lambda^* b^\dagger b = a\rho^* b^\dagger b = (a\rho^*)b$ shows that λ^* and ρ^* are linked.
Thus, for all $(\lambda, \rho) \in \Omega(S)$, (λ^*, ρ^*), $(\lambda^\dagger, \rho^\dagger)$ are members of $\Omega(S)$. More details on this can be found in Guo and Guo (2000) and Guo and Shum (2003)
Theorem 3.3.1 Fountain and Lawson (1985). The translational hull of a type A semigroup is type$A$.

Category Theory
Preliminaries
Category can be viewed in two versions which are indeed implicitly the same. Namely:
The object – morphism version of category and The generalized monoid version of category.
The Object – Morphism Version of Category
According to Asibong-Ibe (1993), category consists of

- a class of objects (usually denoted by $\boldsymbol{C}$- obj)
- a set of morphisms between the objects in $\boldsymbol{C}$ which are denoted by $hom_{\boldsymbol{C}}(A, B)$ or simply $hom(A, B)$ for morphisms between A and B, satisfying the following conditions:
  i. for any set of objects $A, B, C \in \boldsymbol{C}$, the $\boldsymbol{C}$-morphisms $f \in hom(A, B)$, $g \in hom(B, C)$ imply $g \circ f \in hom(A, C)$
  ii. for each object A, an identity morphism $1_A \in hom(A, A)$
  iii. if $f \in hom(A, B)$, $g \in hom(B, C)$ and $h \in hom(C, D)$, then $h\circ(g\circ f) = (h \circ g)\circ f \in hom(A, D)$
  iv. for every object A, $1_A \in hom(A, A)$ and $f \circ 1_A = f$, $1_B \circ g = g$, for every $f, g \in hom(A, B)$.
  v. every distinct pair of $\boldsymbol{C}$- objects has distinct set of morphisms. That is, if $(A, B) \neq (C, D)$, then $hom(A, B) \cap hom(C, D) = \emptyset$.

So, in a category, there must be a class consisting of systems of the same type, referred to as $objects$ and between any pair of objects A and B in the class, there must arrows $f: A \to B$ and each arrow is a structure preserving map referred to as morphism.
4.1.2 Subcategory
Let $\boldsymbol{D}$ be a subclass of a category $\boldsymbol{C}$ such that each object in $\boldsymbol{D}$ is also a $\boldsymbol{C}$- object. Then $\boldsymbol{D}$ is a subcategory if

i. for any pair of objects A, B in $\boldsymbol{D}$, each morphism $f: A \to B$ in $\boldsymbol{D}$ is also a morphism in $\boldsymbol{C}$
ii. each object in $\boldsymbol{D}$ has an identity morphisms in $\boldsymbol{D}$ and
iii. $\boldsymbol{D}$ contains the product of its morphisms. That is, the products of $\boldsymbol{D}$-morphisms $f: A \to B$ and $g: B \to C$ which is $g\circ f: A \to C$ is also a $\boldsymbol{D}$-morphism.

The Generalized Monoid Version of Category

Let C be a class and $'' \cdot ''$ be a partial binary operation on C. For $x, y \in C$, we write $\exists x \cdot y$ if $x \cdot y \in C$. An element $e \in C$ is called an idempotent if $\exists e \cdot e$ and $e \cdot e = e$. The idempotents $e \in C$ which satisfy the conditions that for $x \in C$, $\exists e \cdot x \Rightarrow e \cdot x = x$ and $\exists x \cdot e \Rightarrow x \cdot e = x$, are called the identities of C. We denote the set identities of C by C_o.

The pair $(C, \cdot)$ is called a category if the following hold:

i. $\exists x \cdot (y \cdot z) \Leftrightarrow \exists (x \cdot y) \cdot z$ and in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ………….(Ai)

ii. $\exists x \cdot (y \cdot z) \Leftrightarrow \exists x \cdot y$ and $\exists y \cdot z$ ……………….(Aii)

iii. $\forall x \in C$, there exist unique identities $\boldsymbol{d}(x), \boldsymbol{r}(x) \in C_o$ such that $\exists \boldsymbol{d}(x) \cdot x$ and $\exists x \cdot \boldsymbol{r}(x)$ …….(Aiii)

Whenever the partial multiplication in category $(C, \cdot)$ is clear, we simply refer to category C. The identity $\boldsymbol{d}(x)$ is called the domain of x and the identity $\boldsymbol{r}(x)$ is called the range of x. Since $\boldsymbol{d}(x), \boldsymbol{r}(x) \in C_o$, $\boldsymbol{d}(x) \cdot x = x$ and $x \cdot \boldsymbol{r}(x) = x$. Thus, for any identity e, $\boldsymbol{d}(e) = \boldsymbol{r}(e) = e$.

Lemma 4.1.4 Lawson (1991): Let $(C, \cdot)$ be a category with $x, y \in C$.

i. $\exists x \cdot y \Leftrightarrow \boldsymbol{r}(x) = \boldsymbol{d}(y)$

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ii. If $\exists x \cdot y$, then $\boldsymbol{d}(x \cdot y) = \boldsymbol{d}(x)$ and $\boldsymbol{r}(x \cdot y) = \boldsymbol{r}(y)$.

Let $(C,\cdot)$ be a category. For $,f \in C_o$, we define the set $mor(e,f)$ by:

$mor(e,f) = \{x \in C: \boldsymbol{d}(x) = e, \boldsymbol{r}(x) = f\}$.

When $e = f$, $mor(e,f)$ is a monoid. To see this, for $\in mor(e,e)$, $e \cdot x = \boldsymbol{d}(x) \cdot x = x,\ x \cdot e = \boldsymbol{r}(x) = x$. Therefore, e is the identity in $mor(e,e)$. Let $x, y \in mor(e,e)$. Then, $\boldsymbol{d}(x \cdot y) = \boldsymbol{d}(x) = e$ and $\boldsymbol{r}(x \cdot y) = \boldsymbol{r}(y) = e$. Therefore, $x \cdot y \in mor(e,e)$. It then follows that $\exists x \cdot (y \cdot z)$ and $\exists (x \cdot y) \cdot z$, $\forall\, x, y, z \in mor(e,e)$, and since $mor(e,e) \in C$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

$mor(e,e)$ is called the $local\ submonoid$ of C at e. Thus, category is regarded as a generalization of a monoid. A $unipotent\ category$ is a category in which every local submonoid contains only one idempotent.

Lemma 4.1.5 Lawson (1999): Let $(C,\cdot,\leq)$ be an ordered category and suppose that $a \in C$ and $e \in C_o$. If $a \leq e$, then $a \in C_o$.
Consequently, in an ordered category $(C,\cdot,\leq)$, if the greatest lower bound ($the\ meet$) of two identities – e, f , denoted by $e \wedge f$ (with respect to $\leq$) exists, then it is an identity.

Functor

Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. A function $\phi: \boldsymbol{C} \to \boldsymbol{D}$ is called a $functor$ if it satisfies the following conditions:

i. If $\exists a \cdot b$ in $\boldsymbol{C}$, then $\exists a\phi \cdot b\phi$ in $\boldsymbol{D}$ and
ii. $a\phi \cdot b\phi = (a \cdot b)\phi$

A functor $\phi: \boldsymbol{C} \to \boldsymbol{D}$ is called an $ordered\ functor$ (or $order\ preserving\ functor$) if $a \leq b$ in $\boldsymbol{C}$, then $a\phi \leq b\phi$ in $\boldsymbol{D}$.

Construction of a Category from an Inverse Semigroup

This construction is done in analogy with Lawson (1991)'s construction of inductive category from a restriction semigroup.
Given an inverse semigroup S_1, we define a product in S_1 by

$$a \cdot b = \begin{cases} ab & if\ \ a^{-1}a = bb^{-1} \\ undefined, & otherwise \end{cases} \qquad a, b \in S_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(Gi)}$$

Theorem 4.2.1: Let S_1 be an inverse semigroup with the natural partial order $\leq$. Then $(S_1,\cdot,\leq) = \boldsymbol{C(S_1)}$ is a category with $\boldsymbol{C(S_1)}_o = E(S_1)$, $\boldsymbol{d}(a) = aa^{-1}$, $\boldsymbol{r}(a) = a^{-1}a$, $\forall a \in S_1$, where $'' \cdot ''$ is the product defined in (Gi).
Proof: Assuming e is an identity in $(S_1,\cdot)$such that $\exists e \cdot x$ for $x \in S_1$. Then, by the definition of $'' \cdot ''$, $e = xx^{-1}$. Similarly, if f is an identity in $(S_1,\cdot)$ such that $\exists x \cdot f$ for $x \in S_1$. Then $f = x^{-1}x$. Thus, idempotents in S_1 are the identities in $(S_1,\cdot)$. $xx^{-1} \cdot x$ exists since $(xx^{-1})^{-1}(xx^{-1}) = xx^{-1}$. Of course, $xx^{-1} \cdot x = x$ and by uniqueness of $\boldsymbol{d}(x)$, $xx^{-1} = \boldsymbol{d}(x)$. Similarly, $x^{-1}x = \boldsymbol{r}(x)$.
Next, we show that $(\forall x, y, z \in S_1)\ \exists x \cdot (y \cdot z) \Leftrightarrow \exists (x \cdot y) \cdot z$ and that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $\exists x \cdot y$; $\exists y \cdot z$.
Assuming $\exists x \cdot (y \cdot z)$, then $x^{-1}x = (y \cdot z)(y \cdot z)^{-1}$
But $(y \cdot z) = yz$ such that $y^{-1}y = zz^{-1}$
Therefore, $\exists x \cdot (y \cdot z) \Rightarrow x^{-1}x = (yz)(yz)^{-1}$ and $y^{-1}y = zz^{-1}$
So that $x^{-1}x = (yz)(yz)^{-1} = yzz^{-1}y^{-1} = yzz^{-1}zz^{-1}y^{-1} = yy^{-1}yy^{-1}yy^{-1}$ (since $y^{-1}y = zz^{-1}$) $= yy^{-1}$
Therefore, $\exists x \cdot y$. Similarly, $\exists y \cdot z$.
Again, $x \cdot (y \cdot z) = xyz$ such that $x^{-1}x = (yz)(yz)^{-1}$ and $y^{-1}y = zz^{-1}$. But $x^{-1}x = (yz)(yz)^{-1} = yy^{-1}$.
Therefore, $x \cdot (y \cdot z) = xyz$ such that $x^{-1}x = yy^{-1}$; $y^{-1}y = zz^{-1}$.
On the other hand, $(x \cdot y) \cdot z = xyz$ such that $(xy)^{-1}(xy) = zz^{-1}$ and $x^{-1}x = yy^{-1}$.
But $(xy)^{-1}(xy) = y^{-1}x^{-1}xy = y^{-1}x^{-1}xx^{-1}xy = y^{-1}yy^{-1}yy^{-1}y = y^{-1}y$.
So that $(x \cdot y) \cdot z = xyz$ such that $y^{-1}y = zz^{-1}$; $x^{-1}x = yy^{-1}$
Thus, $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
Hence, $(S_1,\cdot)$ is a category. We denote by $\boldsymbol{C(S_1)}$ this category associated with an inverse semigroup S_1, and the set of identities of $\boldsymbol{C(S_1)}$ by $\boldsymbol{C(S_1)}_o$

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Corollary 4.2.2: Let S_1 be an inverse semigroup with the natural partial order $\leq$. Then $(S_1, \cdot, \leq) = \boldsymbol{C(S_1)}$ is an inductive category with $\boldsymbol{C(S_1)}_o = E(S_1)$, $\boldsymbol{d}(a) = aa^{-1}$, $\boldsymbol{r}(a) = a^{-1}a$, $\forall a \in S_1$, where $'' \cdot ''$ is the product defined in (Gi) above

Construction of a Category from a Type A Semigroup

In a very similar fashion as that of inverse semigroup, a category from a type A semigroup is constructed as follows: Let S be a type A semigroup and define a product in S by

$$a \cdot b = \begin{cases} ab \quad if \;\; a^* = b^\dagger \\ undefined, \;\; otherwise \end{cases} \qquad a, b \in S_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(Gii)}$$

Theorem 4.3.1: Let S be a type A semigroup with the natural partial order $\leq$. Then $(S, \cdot, \leq) = \boldsymbol{C(S)}$ is a category with $\boldsymbol{C(S)}_o = E(S)$, $\boldsymbol{d}(a) = a^\dagger$, $\boldsymbol{r}(a) = a^*$, $\forall a \in S$, where $'' \cdot ''$ is the product defined in (Gii).

Proof: Assuming e is an identity in $(S, \cdot)$ such that $\exists e \cdot x$ for $x \in S$. Then $e = x^\dagger$. Similarly, if f is an identity in $(S, \cdot)$ such that $\exists x \cdot f$ for $x \in S$. Then $f = x^*$. Thus, idempotents in S are the identities in $(S, \cdot)$. $x^\dagger \cdot x$ exists since $(x^\dagger)^* = x^\dagger$. Of course, $x^\dagger \cdot x = x$ and by uniqueness of $\boldsymbol{d}(x)$, $x^\dagger = \boldsymbol{d}(x)$. Similarly, $x^* = \boldsymbol{r}(x)$.

Now, suppose $\exists x \cdot (y \cdot z)$. That is $x^* = (yz)^\dagger$ and $y^* = z^\dagger$. So that $x^* = (yz^\dagger)^\dagger = (yy^*)^\dagger = y^\dagger$. So $\exists x \cdot (y \cdot z) \Rightarrow y^* = z^\dagger; x^* = y^\dagger$. But $(xy)^* = (x^*y)^* = (y^\dagger y)^* = y^* = z^\dagger$. So that $\exists x \cdot (y \cdot z) \Leftrightarrow (xy)^* = z^\dagger; x^* = y^\dagger \Leftrightarrow \exists (x \cdot y) \cdot z$. Hence, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Moreover, $\exists x \cdot (y \cdot z) \Rightarrow x^* = y^\dagger; y^* = z^\dagger \Rightarrow \exists x \cdot y\,; \exists y \cdot z$.

Hence, $(S, \cdot)$ is a category. We denote by $\boldsymbol{C(S)}$ this category associated with type A semigroup S, and the set of identities of $\boldsymbol{C(S)}$ by $\boldsymbol{C(S)}_o$.

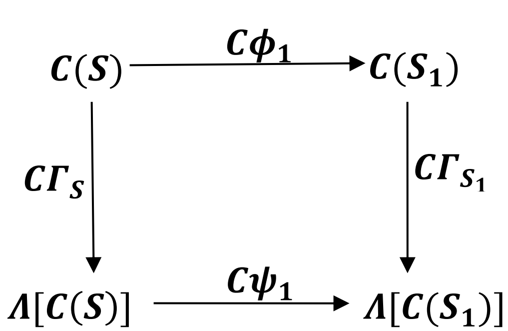
Results

Using theorem 2.2.2 above, Fountain (1979)**:** obtained an embedding of type$A$ semigroup into an inverse semigroup. Offor et al. (2018) extended the representation to the translational hull of type A semigroup. Now, our effort in this section is to further extend the representation to the category of the translational hull of type A monoid.

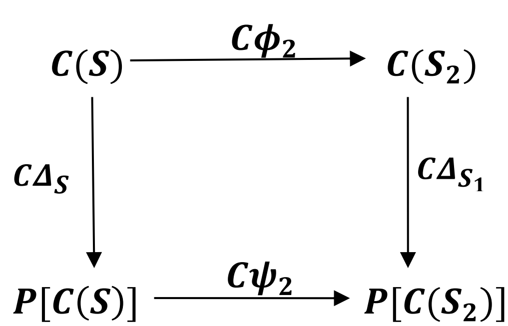
Now, we recall from section 3.1 that for each element a in a semigroup S whose translational hull $\Omega(S)$ is considered, there is a linked pair (λ_a, ρ_a) within $\Omega(S)$ defined by $\lambda_a x = ax$ and $x\rho_a = xa$, and it is called the *inner part* of $\Omega(S)$. The product is defined by $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$. $a \mapsto (\lambda_a, \rho_a)$ is a map of S into $\Omega(S)$ is denoted by Π_S. $\quad \Pi_S(S) = \{(\lambda_a, \rho_a) \,|\, a \in S, \lambda_a x = ax, x\rho_a = xa, \forall x \in S\}$

We let $\boldsymbol{C\Gamma_S}: a \mapsto \lambda_a$, and $\boldsymbol{C\Gamma(S)} = \{\lambda_a : a \in \boldsymbol{C(S)}\}$. We also let $\boldsymbol{C\Delta_S}: a \mapsto \rho_a$ and $\boldsymbol{C\Delta(S)} = \{\rho_a : a \in \boldsymbol{C(S)}\}$.

Theorem 5.1: Given the category of a type$A$ monoid $\boldsymbol{C(S)}$, there are inverse semigroups categories $\boldsymbol{C(S_1)}$, $\boldsymbol{C(S_2)}$, and embeddings $\boldsymbol{C\phi_1}: \boldsymbol{C(S)} \to \boldsymbol{C(S_1)}$, $\boldsymbol{C\phi_2}: (\boldsymbol{S}) \to (\boldsymbol{S_2})$, such that $\boldsymbol{C\phi_1} a^* = (\boldsymbol{C\phi_1} a)^* = (\boldsymbol{C\phi_1} a)^{-1}(\boldsymbol{C\phi_1} a)$, $\boldsymbol{C\phi_2} a^\dagger = (\boldsymbol{C\phi_2} a)^\dagger = (\boldsymbol{C\phi_2} a)(\boldsymbol{C\phi_2} a)^{-1}$, and there are also embeddings $\boldsymbol{C\psi_1}: \boldsymbol{\Lambda}[\boldsymbol{C(S)}] \to \boldsymbol{\Lambda}[(\boldsymbol{S_1})]$, $\boldsymbol{C\psi_2}: \boldsymbol{P}[\boldsymbol{C(S)}] \to \boldsymbol{P}[(\boldsymbol{S_2})]$ such that each of the diagrams



(i)



(ii)

commutes

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and $\boldsymbol{C\psi_1}(\lambda^*) = [\boldsymbol{C\psi_1}(\lambda)]^* = [\boldsymbol{C\psi_1}(\lambda)]^{-1}\, \boldsymbol{C\psi_1}(\lambda)$, $\quad \boldsymbol{C\psi_2}(\rho^\dagger) = [\boldsymbol{C\psi_2}(\rho)]^\dagger = \boldsymbol{C\psi_2}(\rho)[\boldsymbol{C\psi_2}(\rho)]^{-1}$.

We prove this theorem through the following propositions, lemmas and corollaries. Diagram (i) is dual to diagram (ii) and therefore every fact established about diagram (i) applies in dual manner to diagram (ii).

Proposition 5.2: Given the category of type$A$ monoid $\boldsymbol{C}(\boldsymbol{S})$, there are inverse semigroup categories $\boldsymbol{C}(\boldsymbol{S_1})$, $\boldsymbol{C}(\boldsymbol{S_2})$ and embeddings $\boldsymbol{C}(\boldsymbol{\phi_1})\colon \boldsymbol{S} \to \boldsymbol{S_1}$, $\boldsymbol{C}(\boldsymbol{\phi_2})\colon \boldsymbol{S} \to \boldsymbol{S_2}$, such that $\boldsymbol{\phi_1}a^* = (\boldsymbol{\phi_1}a)^* = (\boldsymbol{\phi_1}a)^{-1}(\boldsymbol{\phi_1}a)$, $\boldsymbol{\phi_2}a^\dagger = (\boldsymbol{\phi_2}a)^\dagger = (\boldsymbol{\phi_2}a)(\boldsymbol{\phi_2}a)^{-1}$.

Proof:

Let $\boldsymbol{C}(\boldsymbol{S})$ be a category of type$A$ monoid. To start with, we need to establish the existence of the categories of inverse semigroup(s) $\boldsymbol{C}(\boldsymbol{S_1})$ [and $\boldsymbol{C}(\boldsymbol{S_2})$].

For each $a \in \boldsymbol{C}(\boldsymbol{S})$, $(a^*, a) \in \mathcal{L}^*$ since $\boldsymbol{C}(\boldsymbol{S})$ is abundant.

Define a map $\eta_a\colon a^*\boldsymbol{C}(\boldsymbol{S}) \to a\boldsymbol{C}(\boldsymbol{S})$ defined by $\eta_a(a^*s) = as$, $s \in \boldsymbol{C}(\boldsymbol{S})$.

Let $a^*s' = a^*s''$, $s', s'' \in \boldsymbol{C}(\boldsymbol{S})$.

$\eta_a(a^*s') = as' = aa^*s'$ $\quad$ [a^* is a right identity to a]

$= aa^*s'' = as'' = \eta_a(a^*s'')$. Thus, η_a is well defined.

Let $\eta_a(a^*s_1) = \eta_a(a^*s_2)$, $s_1, s_2 \in \boldsymbol{C}(\boldsymbol{S})$. This implies that $as_1 = as_2$. Since $(a^*, a) \in \mathcal{L}^*$, $as_1 = as_2$ implies that $a^*s_1 = a^*s_2$. Thus, η_a is one – one and therefore, a member of the *symmetric inverse semigroup* $\mathfrak{T}_{\boldsymbol{C}(\boldsymbol{S})}$ on $\boldsymbol{C}(\boldsymbol{S})$. So, $\mathfrak{T}_{\boldsymbol{C}(\boldsymbol{S})}$ becomes the $\boldsymbol{C}(\boldsymbol{S_1})$.

η_a is as well surjective since $\forall as \in a\boldsymbol{C}(\boldsymbol{S})$, $as = aa^*s = \eta_a(a^*s)$, which implies that every element $as \in a\boldsymbol{C}(\boldsymbol{S})$ has a pre – image $a^*s \in a^*\boldsymbol{C}(\boldsymbol{S})$.

Thus, η_a is a bijection.

Hence, $\forall a \in \boldsymbol{C}(\boldsymbol{S})$, there is a bijection $\eta_a\colon a^*\boldsymbol{C}(\boldsymbol{S}) \to a\boldsymbol{C}(\boldsymbol{S})$ defined by $\eta_a(a^*s) = as$, [$s \in \boldsymbol{C}(\boldsymbol{S})$] which maps a^* to a.

Now, define a map $\boldsymbol{C}(\boldsymbol{\phi_1})\colon \boldsymbol{C}(\boldsymbol{S}) \to \boldsymbol{C}(\boldsymbol{S_1})$ by $(\boldsymbol{C}(\boldsymbol{\phi_1}))(a) = \eta_a$, [$a \in \boldsymbol{C}(\boldsymbol{S})$]

For $a, b \in \boldsymbol{C}(\boldsymbol{S})$, the domain of $\eta_b\eta_a$ is $\eta_a^{-1}(b^*\boldsymbol{C}(\boldsymbol{S}) \cap a\boldsymbol{C}(\boldsymbol{S}))$

$\eta_a^{-1}(b^*\boldsymbol{C}(\boldsymbol{S}) \cap a\boldsymbol{C}(\boldsymbol{S})) = \eta_a^{-1}(b^*a\boldsymbol{C}(\boldsymbol{S}))$ [since for the category of a type A semigroup $\boldsymbol{C}(\boldsymbol{S})$, $e\boldsymbol{C}(\boldsymbol{S}) \cap a\boldsymbol{C}(\boldsymbol{S}) = ea\boldsymbol{C}(\boldsymbol{S})$, $(\forall a \in \boldsymbol{C}(\boldsymbol{S}))(\forall e \in \boldsymbol{C}(\boldsymbol{S})_o)$].

$= \eta_a^{-1}[a(b^*a)^*\boldsymbol{C}(\boldsymbol{S})] = \eta_a^{-1}\eta_a[(b^*a)^*\boldsymbol{C}(\boldsymbol{S})] = (b^*a)^*\boldsymbol{C}(\boldsymbol{S})$.

Now, since $ba\mathcal{L}^*(ba)^* = (b^*a)^*$, $dom(\eta_{ba}) = (b^*a)^*\boldsymbol{C}(\boldsymbol{S}) = dom(\eta_b\eta_a)$.

For $(b^*a)^*s \in (b^*a)^*\boldsymbol{C}(\boldsymbol{S}) = dom(\eta_{ba})$

$\eta_{ba}[(b^*a)^*s] = bas = bb^*as = \eta_b b^*as = \eta_b a(b^*a)^*s\ \eta_b\eta_a[(b^*a)^*s]$

Thus, $\eta_{ba} = \eta_b\eta_a\ \ \forall a, b \in \boldsymbol{C}(\boldsymbol{S})$.

Hence, $\forall a, b \in \boldsymbol{C}(\boldsymbol{S})$, $(\boldsymbol{C}(\boldsymbol{\phi_1}))ba = \eta_{ba} = \eta_b\eta_a = [\boldsymbol{C}(\boldsymbol{\phi_1})]b[\boldsymbol{C}(\boldsymbol{\phi_1})]a$.

Therefore, $\boldsymbol{C}(\boldsymbol{\phi_1})$ is a homomorphism.

Let $(\boldsymbol{C}(\boldsymbol{\phi_1}))a = (\boldsymbol{C}(\boldsymbol{\phi_1}))b$. This implies that $\eta_a = \eta_b$. That is, $dom\,\eta_a = dom\,\eta_b = e\boldsymbol{C}(\boldsymbol{S})$ (say).

Then, $a = \eta_a(e) = \eta_b(e) = b$.

Thus, $\boldsymbol{C}(\boldsymbol{\phi_1})$ is one – one and hence an embedding.

Now, we want to establish that $\boldsymbol{C}(\boldsymbol{\phi_1})a^* = (\boldsymbol{C}(\boldsymbol{\phi_1})a)^* = (\boldsymbol{C}(\boldsymbol{\phi_1})a)^{-1}(\boldsymbol{C}(\boldsymbol{\phi_1})a)$, $(\forall a \in \boldsymbol{C}(\boldsymbol{S}))$

Since $\boldsymbol{C}(\boldsymbol{S_1})$ is regular, $\mathcal{L}(\boldsymbol{C}(\boldsymbol{S_1})) = \mathcal{L}^*(\boldsymbol{C}(\boldsymbol{S_1}))$.

So that for $a \in \boldsymbol{C}(\boldsymbol{S})$, $[\boldsymbol{C}(\boldsymbol{\phi_1})]a \in \boldsymbol{C}(\boldsymbol{S_1})$, $[\boldsymbol{C}(\boldsymbol{\phi_1})a]^* = (\boldsymbol{C}(\boldsymbol{\phi_1})a)^{-1}(\boldsymbol{C}(\boldsymbol{\phi_1})a) \in \mathcal{L}_{\boldsymbol{C}(\boldsymbol{\phi_1})a}$ since the idempotent must be unique.

We show next that $\boldsymbol{C}(\boldsymbol{\phi_1})a^* = [\boldsymbol{C}(\boldsymbol{\phi_1})a]^*$.

Let $a, x, y \in \boldsymbol{C}(\boldsymbol{S})$. $[\boldsymbol{C}(\boldsymbol{\phi_1})a\,\boldsymbol{C}(\boldsymbol{\phi_1})x = \boldsymbol{C}(\boldsymbol{\phi_1})a\boldsymbol{C}(\boldsymbol{\phi_1})y] \Leftrightarrow [\boldsymbol{C}(\boldsymbol{\phi_1})(ax) = \boldsymbol{C}(\boldsymbol{\phi_1})(ay)]$

$\Leftrightarrow [ax = ay] \Leftrightarrow [a^*x = a^*y] \Leftrightarrow [\boldsymbol{C}(\boldsymbol{\phi_1})(a^*x) = \boldsymbol{C}(\boldsymbol{\phi_1})(a^*y)]$

$\Leftrightarrow [\boldsymbol{C}(\boldsymbol{\phi_1})(a^*)\boldsymbol{C}(\boldsymbol{\phi_1})(x) = \boldsymbol{C}(\boldsymbol{\phi_1})(a^*)\,\boldsymbol{C}(\boldsymbol{\phi_1})(y)] \Leftrightarrow \boldsymbol{C}(\boldsymbol{\phi_1})(a)\mathcal{L}^*\,\boldsymbol{C}(\boldsymbol{\phi_1})(a^*)$.

Since $\boldsymbol{C}(\boldsymbol{\phi_1})$ is a homomorphism, $\boldsymbol{C}(\boldsymbol{\phi_1})(a^*)$ is an idempotent in $\mathfrak{T}_{\boldsymbol{C}(\boldsymbol{S})}$ and since idempotent in $\mathcal{L}_{\boldsymbol{C}(\boldsymbol{\phi_1})a}$ must be unique, $\boldsymbol{C}(\boldsymbol{\phi_1})(a^*) = (\boldsymbol{C}(\boldsymbol{\phi_1})a)^*$.

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We have just shown that given a type$A$ monoid $\boldsymbol{C(S)}$, there is an inverse semigroup $\boldsymbol{C(S_1)}$ and an embedding $\boldsymbol{C(\phi_1)}: \boldsymbol{C(S)} \to \boldsymbol{C(S_1)}$ such that $\boldsymbol{C(\phi_1)}a^* = (\boldsymbol{C(\phi_1)}a)^* = (\boldsymbol{C(\phi_1)}a)^{-1}(\boldsymbol{C(\phi_1)}a)$.
Referring to $\mathcal{R}^*$ (instead of $\mathcal{L}^*$), $\mathfrak{T}_{\boldsymbol{C(S)}}$ becomes our $\boldsymbol{C(S_2)}$, and carrying out the dual argument gives the second part of the result –
Given a type$A$ monoid $\boldsymbol{C(S)}$, there is a category of an inverse semigroup $\boldsymbol{C(S_2)}$ and an embedding $\boldsymbol{C(\phi_2)}: \boldsymbol{C(S)} \to \boldsymbol{C(S_2)}$ such that $\boldsymbol{C(\phi_2)}a^\dagger = (\boldsymbol{C(\phi_2)}a)^\dagger = (\boldsymbol{C(\phi_2)}a)(\boldsymbol{C(\phi_2)}a)^{-1}$.

Lemma 5.3: For an inverse semigroup $\boldsymbol{C(S_1)}$, $\boldsymbol{C(\Gamma)}: a \mapsto \lambda_a$ is an isomorphism from $\boldsymbol{C(S_1)}$ onto $\boldsymbol{C(\Gamma)}[\boldsymbol{C(S_1)}]$.
Proof:
For $a, b \in \boldsymbol{C(S_1)}$, $\boldsymbol{C(\Gamma)}[ab] = \lambda_{ab} = \lambda_a\lambda_b = \boldsymbol{C(\Gamma)}(a)\boldsymbol{C(\Gamma)}(b)$. Therefore, $\boldsymbol{C(\Gamma)}$ is a homomorphism.
Let $\lambda_a = \lambda_b$.
$a = aa^{-1}a = \lambda_a a^{-1}a = \lambda_b a^{-1}a = ba^{-1}a \le b$.
Similarly, $b \le a$, and therefore $a = b$.
Thus, $\boldsymbol{C(\Gamma)}$ is injective. It is also an onto map since $\forall\lambda_a \in \boldsymbol{C(\Gamma)}[\boldsymbol{C(S_1)}]$, $\exists a \in \boldsymbol{C(S)}$ with $a \mapsto \lambda_a$.

Lemma 5.4: For a type$A$ semigroup $\boldsymbol{C(S)}$, $\boldsymbol{C(\Gamma)}: a \mapsto \lambda_a$ is an isomorphism from $\boldsymbol{C(S)}$ onto $\boldsymbol{C(\Gamma)}[\boldsymbol{C(S)}]$.
For $a, b \in \boldsymbol{C(S_1)}$, $\boldsymbol{C(\Gamma)}[ab] = \lambda_{ab} = \lambda_a\lambda_b = \boldsymbol{C(\Gamma)}(a)\boldsymbol{C(\Gamma)}(b)$. Therefore, $\boldsymbol{C(\Gamma)}$ is a homomorphism.
Let $\lambda_a = \lambda_b$.
$a = aa^* = \lambda_a a^* = \lambda_b a^* = ba^* \le b$. Similarly, $b \le a$, and therefore $a = b$.
Thus, $\boldsymbol{C(\Gamma)}$ is injective. It is also an onto map since $\forall\lambda_a \in \boldsymbol{C(\Gamma)}[\boldsymbol{C(S)}]$, $\exists a \in \boldsymbol{C(S)}$ with $a \mapsto \lambda_a$.

Corollary 5.5: For a category of an inverse semigroup $\boldsymbol{C(S_1)}$, $\boldsymbol{C(\Delta_{S_1})}: a \mapsto \rho_a$ is an isomorphism from $\boldsymbol{C(S_1)}$ onto $\boldsymbol{C(\Delta)}[\boldsymbol{C(S_1)}]$. Similarly, for a category of a type$A$ semigroup $\boldsymbol{C(S)}$, $\boldsymbol{C(\Delta_S)}: a \mapsto \rho_a$ is an isomorphism from $\boldsymbol{C(S)}$ onto $\boldsymbol{C(\Delta)}[\boldsymbol{C(S)}]$.

Proposition 5.6: Given a category of a type$A$ monoid $\boldsymbol{C(S)}$, there are categories of inverse semigroups $\boldsymbol{C(S_1)}, \boldsymbol{C(S_2)}$, and embeddings $\boldsymbol{C\psi_1}: \boldsymbol{\Lambda}[\boldsymbol{C(S)}] \to \boldsymbol{\Lambda}[\boldsymbol{C(S_1)}]$, $\boldsymbol{C\psi_2}: \boldsymbol{P}[\boldsymbol{C(S)}] \to \boldsymbol{P}[\boldsymbol{C(S_2)}]$ such that $\boldsymbol{C\psi_1}(\lambda^*) = [\boldsymbol{C\psi_1}(\lambda)]^* = [\boldsymbol{C\psi_1}(\lambda)]^{-1}\, \boldsymbol{C\psi_1}(\lambda)$, $\boldsymbol{C\psi_2}(\rho^\dagger) = [\boldsymbol{C\psi_2}(\rho)]^\dagger = \boldsymbol{C\psi_2}(\rho)[\boldsymbol{C\psi_2}(\rho)]^{-1}$.
Proof:
Let us denote the symmetric inverse semigroup on $\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$ by $\mathfrak{T}_{\boldsymbol{\Lambda}[\boldsymbol{C(S)}]}$. For each $\lambda \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, we define a map

$$\theta_\lambda: \lambda^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}] \to \lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}] \text{ by } \theta_\lambda(\lambda^*\lambda_1) = \lambda\lambda_1, \quad \lambda_1 \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}].$$

We show that θ_λ is one-one.
Let $\theta_\lambda(\lambda^*\lambda_1) = \theta_\lambda(\lambda^*\lambda_2)$, $\lambda_1, \lambda_2 \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$. This implies that $\lambda\lambda_1 = \lambda\lambda_2$.
Since $\lambda\mathcal{L}^*\lambda^*$, $\lambda\lambda_1 = \lambda\lambda_2 \Leftrightarrow \lambda^*\lambda_1 = \lambda^*\lambda_2$.
Thus, θ_λ is one-one and $\theta_\lambda \in \mathfrak{T}_{\boldsymbol{\Lambda}[\boldsymbol{C(S)}]}$. So that we take $\Lambda(S_1)$ to be $\mathfrak{T}_{\boldsymbol{\Lambda}[\boldsymbol{C(S)}]}$
Evidently, θ_λ is surjective since $\forall\lambda\lambda_1 \in \lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, $\lambda\lambda_1 = \lambda\lambda^*\lambda_1 = \theta_\lambda\lambda^*\lambda_1$. So that every $\lambda\lambda_1 \in \lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$ has a pre-image $\lambda^*\lambda_1$ in $\lambda^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$.
Hence, $\forall\lambda \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, there is a bijection $\theta_\lambda: \lambda^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}] \to \lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$ defined by $\theta_\lambda(\lambda^*\lambda_1) = \lambda\lambda_1$, $\lambda_1 \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, which maps λ^* to λ.
Now, define the map $\boldsymbol{C\psi_1}: \boldsymbol{\Lambda}[\boldsymbol{C(S)}] \to \boldsymbol{\Lambda}[\boldsymbol{C(S_1)}]$ by $\boldsymbol{C\psi_1}(\lambda) = \theta_\lambda$.
We show that $\boldsymbol{C\psi_1}$ is a homomorphism.
For $\lambda, l \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, the domain of $\theta_l\theta_\lambda$ is $\theta_\lambda^{-1}\big[\lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}] \cap l^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]\big]$ [see Howie 1995, pg 148]
This implies that $dom\ \theta_l\theta_\lambda = \theta_\lambda^{-1}\big[l^*\lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}]\big]$ since for a type A semigroup, $eS \cap aS = eaS$, $(\forall a \in S)$ $(\forall e \in E)$
$\theta_\lambda^{-1}\big[l^*\lambda\boldsymbol{\Lambda}[\boldsymbol{C(S)}]\big] = \theta_\lambda^{-1}\big[\lambda(l^*\lambda)^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]\big] = \theta_\lambda^{-1}\theta_\lambda\big[(l^*\lambda)^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]\big] = (l^*\lambda)^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$
Since $l\lambda\mathcal{L}^*(l\lambda)^* = (l^*\lambda)^*$, $dom\ \theta_{l\lambda} = (l^*\lambda)^*\boldsymbol{\Lambda}[\boldsymbol{C(S)}]$.
Thus, $dom\ \theta_l\theta_\lambda = dom\ \theta_{l\lambda}$.
Moreover, for $(l^*\lambda)^*\lambda_1 \in dom\ \theta_{l\lambda}$
$\theta_{l\lambda}[(l^*\lambda)^*\lambda_1] = l\lambda\lambda_1 = ll^*\lambda\lambda_1 = \theta_l[l^*\lambda\lambda_1] = \theta_l[\lambda(l^*\lambda)^*\lambda_1] = \theta_l\theta_\lambda[(l^*\lambda)^*\lambda_1]$
Hence, $\theta_l\theta_\lambda = \theta_{l\lambda}$.
Therefore, $\forall\lambda, l \in \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$, $\boldsymbol{C\psi_1}(l\lambda) = \theta_{l\lambda} = \theta_l\theta_\lambda = \boldsymbol{C\psi_1}(l)\boldsymbol{C\psi_1}(\lambda)$. Thus, $\boldsymbol{C\psi_1}$ is a homomorphism.

Let $\boldsymbol{C\psi_1}(l) = \boldsymbol{C\psi_1}(\lambda)$. Then, $\theta_l = \theta_\lambda$. That is, $dom\ \theta_l = dom\ \theta_\lambda = \lambda'\ \boldsymbol{\Lambda}[\boldsymbol{C(S)}]$(say).

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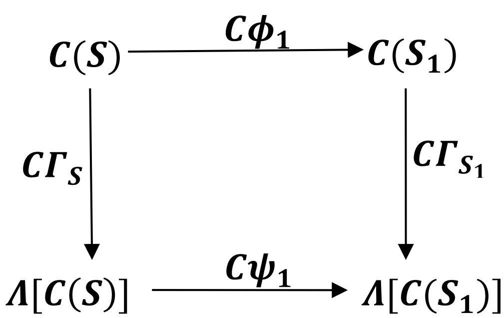
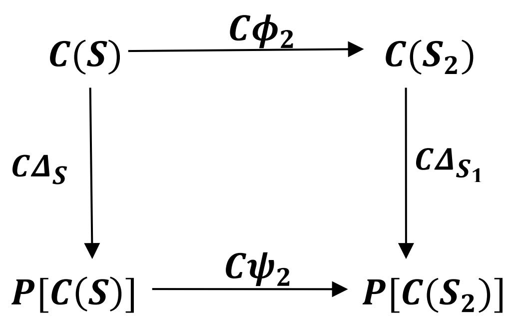
Therefore, $\lambda = \theta_\lambda(\lambda') = \theta_l(\lambda') = l$. Thus, $\boldsymbol{C\psi_1}$ is injective and hence an embedding.
$\mathcal{L}(\boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S_1})]) = \mathcal{L}^*(\boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S_1})])$ since $\boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S_1})]$ is regular. Therefore, for each $\boldsymbol{C\psi_1}(\lambda) \in \boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S_1})]$, $[\boldsymbol{C\psi_2}(\lambda)]^* = [\boldsymbol{C\psi_1}(\lambda)]^{-1}\boldsymbol{C\psi_1}(\lambda) \in \mathcal{L}_{\boldsymbol{C\psi_1}(\lambda)} = \mathcal{L}^*_{\boldsymbol{C\psi_1}(\lambda)}$ since the idempotent must be unique.
Now, we just need to show that $\boldsymbol{C\psi_1}(\lambda^*) = [\boldsymbol{C\psi_1}(\lambda)]^*$
For $\lambda, \lambda', \lambda'' \in \boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S})]$, let $\boldsymbol{C\psi_1}(\lambda)\boldsymbol{C\psi_1}(\lambda') = \boldsymbol{C\psi_1}(\lambda)\boldsymbol{C\psi_1}(\lambda'') \Leftrightarrow \boldsymbol{C\psi_1}(\lambda\lambda') = \boldsymbol{C\psi_1}(\lambda\lambda'') \Leftrightarrow \lambda\lambda' = \lambda\lambda''$
$\Leftrightarrow \lambda^*\lambda' = \lambda^*\lambda'' \Leftrightarrow \boldsymbol{C\psi_1}(\lambda^*)\boldsymbol{C\psi_1}(\lambda') = \boldsymbol{C\psi_1}(\lambda^*)\boldsymbol{C\psi_1}(\lambda'')$. So that $\boldsymbol{C\psi_1}(\lambda)\mathcal{L}^*\boldsymbol{C\psi_1}(\lambda^*)$.
Since $\boldsymbol{C\psi_1}$ is a homomorphism and λ^* an idempotent in $\boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S})]$, $\boldsymbol{C\psi_1}(\lambda^*)$ is an idempotent in $\boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S_1})]$ and since idempotent in $\mathcal{L}_{\boldsymbol{C\psi_1}(\lambda)}$ must be unique, $\boldsymbol{C\psi_1}(\lambda^*) = [\boldsymbol{C\psi_1}(\lambda)]^*$.
By dual argument, it follows that $\boldsymbol{C\psi_2}: \boldsymbol{P}[\boldsymbol{C}(\boldsymbol{S})] \to \boldsymbol{P}[\boldsymbol{C}(\boldsymbol{S_2})]$ is an embedding such that $\boldsymbol{C\psi_2}(\rho^\dagger) = [\boldsymbol{C\psi_2}(\rho)]^\dagger = \boldsymbol{C\psi_2}(\rho)[\boldsymbol{C\psi_2}(\rho)]^{-1}$

Proposition 5.7: Each of the diagrams commutes



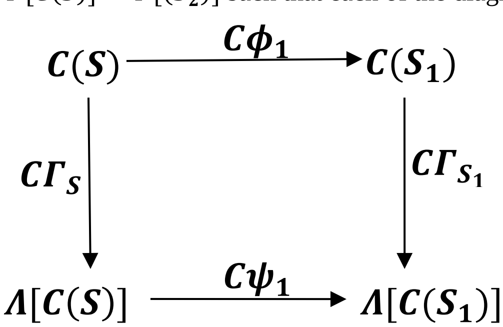
(i)

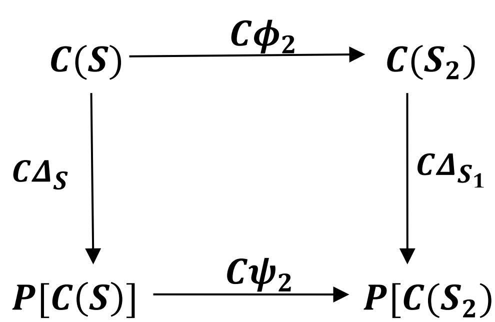


(ii)

Proof: We defined the map $\boldsymbol{C\phi_1}: \boldsymbol{C}(\boldsymbol{S}) \to \boldsymbol{C}(\boldsymbol{S_1})$ by $\boldsymbol{C\phi_1}(a) = \eta_a$. The rest are –
$\boldsymbol{C\Gamma_S}: a \to \lambda_a$, $\boldsymbol{C\Gamma_{S_1}} : \eta_a \to \lambda_{\eta_a}$ and $\boldsymbol{C\psi_1}: \lambda_a \to \theta_{\lambda_a}$
Thus, $\boldsymbol{C\psi_1}[\boldsymbol{C\Gamma_S}(a)] = \theta_{\lambda_a}$ and $\boldsymbol{C\Gamma_{S_1}}[\boldsymbol{C\phi_1}(a)] = \lambda_{\eta_a}$
So, for any $x \in \boldsymbol{C}(\boldsymbol{S})$, $\theta_{\lambda_a}(x) = \lambda_a(x) = ax$
and $\lambda_{\eta_a}(x) = ax = \lambda_a(x) = \theta_{\lambda_a}(x)$
Therefore, $\boldsymbol{C\psi_1 C\Gamma_S} = \boldsymbol{C\Gamma_{S_1}}\,\boldsymbol{C\phi_1}$. Hence, diagram (i) commutes and dually, diagram (ii) commutes.

Conclusion
In this paper, we basically engaged ourselves with the study of translational hulls of type A semigroup and inverse semigroup alongside their categories.
The major result is that given the category of a type$A$ monoid $\boldsymbol{C}(\boldsymbol{S})$, there are inverse semigroups categories $\boldsymbol{C}(\boldsymbol{S_1})$, $\boldsymbol{C}(\boldsymbol{S_2})$, and embeddings $\boldsymbol{C\phi_1}: \boldsymbol{C}(\boldsymbol{S}) \to \boldsymbol{C}(\boldsymbol{S_1})$, $\boldsymbol{C\phi_2}: (\boldsymbol{S}) \to (\boldsymbol{S_2})$, such that $\boldsymbol{C\phi_1}a^* = (\boldsymbol{C\phi_1}a)^* = (\boldsymbol{C\phi_1}a)^{-1}(\boldsymbol{C\phi_1}a)$, $\boldsymbol{C\phi_2}a^\dagger = (\boldsymbol{C\phi_2}a)^\dagger = (\boldsymbol{C\phi_2}a)(\boldsymbol{C\phi_2}a)^{-1}$, and there are also embeddings $\boldsymbol{C\psi_1}: \boldsymbol{\Lambda}[\boldsymbol{C}(\boldsymbol{S})] \to \boldsymbol{\Lambda}[(\boldsymbol{S_1})]$, $\boldsymbol{C\psi_2}: \boldsymbol{P}[\boldsymbol{C}(\boldsymbol{S})] \to \boldsymbol{P}[(\boldsymbol{S_2})]$ such that each of the diagrams





commutes (i) (ii)

and $\boldsymbol{C\psi_1(\lambda^*) = [C\psi_1(\lambda)]^* = [C\psi_1(\lambda)]^{-1} C\psi_1(\lambda)}$, $\quad \boldsymbol{C\psi_2(\rho^\dagger) = [C\psi_2(\rho)]^\dagger = C\psi_2(\rho)[C\psi_2(\rho)]^{-1}}$.

Just as abundant semigroups are analogous to regular semigroups, so are type A semigroups analogous to inverse semigroups. Just as the study of abundant semigroups is guided by the existing results for regular semigroups, study of type A semigroups is guided by the existing results for inverse semigroups. This explains the reason why inverse semigroup is found in most corners of this paper even though our primary target is type A semigroup. Behind most successful results in type A semigroup are existing results in inverse semigroups! Besides, we cannot talk about representation of type A semigroup without talking about the inverse semigroup.

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TABLE OF SYMBOLS

$\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$	Green's relations	
$\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{D}^*, \mathcal{J}^*$	Extended Green's relations	
λ	left translation	lambda
$\Lambda(S)$	The set of the left translations of a semigroup S	$\Lambda =$ capital letter lamda
ρ	right translation	rho
$P(S)$	The set of the right translations of a semigroup S	$P =$ capital letter rho
$\Omega(S)$	the translation hull of a semigroup S	$\Omega =$ omega
(λ_a, ρ_a)	inner part of $\Omega(S)$	
$\Pi_S(S)$	Set of inner parts of $\Omega(S)$	
$\Gamma(S)$	Set of left inner parts of $\Omega(S)$	
Γ_S or simply Γ	an isomorphism from S onto $\Gamma(S)$	$\Gamma =$ gamma
$\varphi \vert A$	The restriction of a function φ to a subset A of its domain	
E_S or $E(S)$	The set of all the idempotents of a semigroup S	
$E_{\Omega(S)}$	The set of all the idempotents of $\Omega(S)$	
Π_S	homomorphism of S into $\Omega(S)$	

a^*	unique idempotent in the $\mathcal{L}^*$	
$a^\dagger$	unique idempotent in the $\mathcal{R}^*$	
$\langle a \rangle$	A set generated by a	
μ	The largest congruence contained in $\mathcal{H}^*$	Greek letter, pronounced Mu
$\mathfrak{I}_X$	symmetric inverse semigroup on set X.	
ι	diagonal relation or identity map	$\iota =$ iota, Greek letter
1_X or ι_X	diagonal relation on X or identity map on X	
$C(S)$	a category constructed from a semigroup S	
$S[C(S)]$	a semigroup constructed from a category $\boldsymbol{C}(\boldsymbol{S})$	
S/α	a factor semigroup	

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