



## Conceptualizing Algebraic Thinking for Undergraduate Mathematics Majors: An Integrated Framework and Pedagogical Implications

\*<sup>1</sup>Amao, F.A., <sup>1</sup>Oladapo, D.I., & <sup>2</sup>Reginal-Ihedike, M.O.

<sup>1</sup>Department of Mathematical Sciences, Adeleke University, Ede, Osun State Nigeria

<sup>2</sup>Department of Science Education, Ignatius Ajuru University of Education, Port Harcourt, Rivers State

\*Corresponding author email: [amao.folake@adelekeuniversity.edu.ng](mailto:amao.folake@adelekeuniversity.edu.ng)

### Abstract

The idea of algebraic thinking is fundamental to undergraduate mathematics, although its definition in relation to mathematics majors is under-theorized. This paper synthesizes contemporary conceptual frameworks in order to propose an integrated multidimensional framework for algebraic thinking among undergraduate mathematics majors. Thematic synthesis was used to examine an integrative review of 35 peer-reviewed sources (2015-2025). Four interconnected dimensions were identified: generalization and abstraction, structural reasoning, representational fluency, and proof based algebraic argumentation. These dimensions draw upon cultural-semiotic theory, APOS theory, commognitive theory and Realistic Mathematics Education. The synthesis highlights four corresponding pedagogical approaches; inquiry-based learning, integration of digital technologies, discourse-rich pedagogy, and contextual problem-solving. Six major challenges are also identified, such as the school-university transition and assessment misalignment. The findings support a multidimensional teaching approach which prepares students to engage in advanced mathematical practice. The suggested framework is theoretical and needs to be empirically proven.

**Keywords:** Algebraic Thinking, Undergraduate Mathematics Education, Structural Reasoning, Representational Fluency, Commognition

### Introduction

Algebraic thinking is widely considered as one of the foundations of mathematical reasoning, which is the basis of the student skills to generalize, abstract, represent and manipulate mathematical structures (Kieran, 2018, 2021; Kaput, 2008/2019). Algebraic thinking in this paper denotes the cognitive and discursive activities of generalization, abstraction, structural representation and reasoning about mathematical systems with special reference to proof based argumentation. At the undergraduate level, mathematics majors encounter algebra not as a set of symbolic procedures to be rehearsed but as a sophisticated domain of structural inquiry encompassing group theory, ring theory, linear algebra, and related fields. The passage from school algebra characterized largely by equation-solving, variable manipulation, and function evaluation to the abstract reasoning demanded by university mathematics is among the most demanding intellectual transitions students face (Tall, 2019; Weber, Lew, & Mejía-Ramos, 2022). Although algebraic thinking is the heart of undergraduate mathematics, the construct is vaguely defined and inconsistently used in tertiary literature. The majority of foundational studies have been carried out in a primary and secondary environment, exploring generalization, pattern recognition, and symbolic reasoning in younger students (Blanton et al., 2015; Carraher & Schliemann, 2018; Radford, 2018). These contributions are fundamental but fail to reflect what undergraduates need to do in order to interact with algebra as axiomatic, proof-oriented field (Mejia-Ramos, Lew, & Weber, 2021; Nardi & Ryve, 2023). Literature has not yet fully theorized the substantive difference between a student who competently factors polynomials and one who reasons fluently about quotient groups. The necessity of a stricter conceptualization at the undergraduate level is supported by the call to pedagogical reform (Rasmussen & Wawro, 2021). Although conventional lecture-based teaching is efficient, it always prioritizes procedural reproduction over the adaptability of structural thinking (Nardi & Ryve, 2023). Such pedagogies as reform-oriented, inquiry-based learning, technology-enhanced instruction, discourse-rich approaches, have proved to be promising but need a clear description of their cognitive and epistemological goals (Laursen & Rasmussen, 2019; Borba, Chiari & Almeida, 2021). This has been accelerated by international policy frameworks such as UNESCO (2021) that have provided momentum by

demanding higher-order thinking capacities. This paper discusses three goals. First, it integrates the recent theoretical studies to put forward a multidimensional conceptualization of algebraic thinking of the undergraduate mathematics majors. Second, it looks at pedagogical implications. Third, it also provides curriculum design, instruction and research recommendations. The vast majority of the conceptions of algebraic thinking were developed to serve students in the lower levels of education, and do not address the specific needs of undergraduate mathematics majors. Theoretical frameworks have seldom been integrated into a unified model for undergraduate thinking in algebra. To the best of our knowledge, no previous synthesis has combined APOS, commognitive, cultural-semiotic and RME frameworks of undergraduate algebra. These gaps have been addressed in this paper.

The study is informed by three research questions:

1. What can be done to conceptualize algebraic thinking as a multidimensional construct suitable to undergraduate mathematics majors?
2. Which theoretical approaches can best shed some light on the dimensions and formation of algebraic thinking at the tertiary level?
3. What are the pedagogical implications of an integrated conceptualization of undergraduate mathematics teaching?

## Background / Theoretical Framework

### The Evolution of Algebraic Thinking as a Construct

Early conceptualizations equated algebraic competence with symbol manipulation and equation-solving (Kieran's procedural-generational view). Kilpatrick, Swafford, and Findell (2001) offered an influential corrective with five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Kaput (2008/2019) argued that algebra comprises two core aspects: (a) generalization and expression of generalizations, and (b) syntactically guided manipulation of formalisms. Carraher and Schliemann (2018) extended this into early algebra, showing young children engage in algebraic reasoning with appropriate support. Kieran (2021) distinguished three interconnected activities: generational (producing expressions representing general relationships), transformational (manipulating symbolic expressions), and global/meta-level (reasoning about structures, constructing proofs). The multidimensional perception of this study is strengthened by recent systematic reviews. Jamil, Rosli, Mahmud, and Hasim (2025) conducted a synthesis of 25 articles (2022-2024) and found cognitive transitions, innovative pedagogies, representational fluency, and developmental alignment. These systematic findings underscore the need for a tertiary-specific model of algebraic thinking.

### Theoretical Frameworks for Algebraic Thinking

#### Cultural-Semiotic Perspectives

Radford (2018, 2022, 2024) advances a cultural-semiotic account: mathematical cognition is embodied, culturally situated, and historically mediated. Algebraic thinking has developed as a result of engagement in historically developed meaning-making systems, mediated by artifacts, as well as signs and symbols. This view has a direct implication on the undergraduates who are expected to appropriate specialized mathematical discourses.

#### APOS Theory

Dubinsky and McDonald (2017) developed Action-Process-Object-Schema theory, positing a progression from actions (external procedures) to processes (internalized actions) to objects (encapsulated processes) to schemas (coherent organizations). In the context of undergraduate algebra, APOS theory explains how a student might first act on symbols (Action), internalize operations (Process), encapsulate them into algebraic structures (Object), and finally integrate these into a coherent understanding of, for example, group or field properties (Schema). In algebraic thinking, APOS maps the trajectory from procedural manipulation to structural conceptualization. Recent APOS research extends to advanced topics: Bilondi and Radmehr (2023) integrated APOS with Tall's three worlds to investigate tree concepts; Vargas and Oktaç (2024) designed a teaching sequence for vector spaces, showing the importance of prerequisite concepts; Borji and Martínez-Planell (2024) contributed to APOS-based tasks linking Riemann sums and double integrals.

#### Commognitive Theory

Sfard (2020) reconceptualizes mathematical thinking as communication. The vocabularies, visual mediators, approved narratives and routines define mathematical discourse. Commognitive conflicts occur when existing discourse of a student, which is influenced by school mathematics, functions on different principles than the target discourse. Recent studies: Molina et al. (2023) investigated how undergraduates conceptualize geometric solids, and found mismatches between students' discursive interactions and the expectations of lecturers; Kontorovich

(2023) identified asymmetrical discourse patterns in student-student interaction. In another study, Karavi, Mali, and Avraamidou (2022) proposed commognition for the study of proof teaching at the university level.

#### Realistic Mathematics Education

Gravemeijer & Doorman (2020) articulate Realistic Mathematics Education (RME): contextual problems as starting points for guided reinvention. Melaibari & Ismail (2023) implemented an RME-based pre-calculus course with 83 freshmen, finding significant positive effects on mathematical thinking, reasoning, and handling of formalism. Representation competency was lowest, suggesting RME needs deliberate scaffolding of representational fluency.

#### Algebraic Thinking in Undergraduate Mathematics

Tall (2019) describes the transition from school to university mathematics as a shift from embodied and symbolic worlds to a formal world of axiomatic definition and deductive proof. Weber, Lew, and Mejía-Ramos (2022) argue that instruction must move students from attending to symbolic form as an end in itself to attending to properties and organizational principles defining algebraic systems. Mejía-Ramos, Lew, and Weber (2021) revealed that the understanding of proofs is closely intertwined with algebraic reasoning, which is proven empirically. The study of Mulligan and Woolcott (2024) identify other understudied areas in early mathematics education, focusing on deep connected understanding, and such includes algebraic habits of mind. Similarly, a diagnostic study of novice mathematics majors (Reyes-Rodríguez & Santos-Trigo, 2024) found that students lacked deductive reasoning skills and tended to generalize from specific cases, highlighting the gap between school-level algebraic manipulation and the structural reasoning required at university.

#### Pedagogical Approaches

**Inquiry-Based Learning (IBL)**- There is a long-term interest in inquiry-based learning (IBL) (Laursen & Rasmussen, 2019; Rasmussen & Wawro, 2021). A quasi-experimental study by Zavala (2023) was carried out at a Predominantly Black Institution, revealing significantly better performance in IBL sections than in traditional lecture. In a flipped linear algebra classroom, Andreasen and Misfeldt (2024) investigated IBL, where they identified enhanced structural reasoning.

**Digital Technology Integration**- It was revealed that digital technology integration builds representational fluency and structural understanding in students (Borba, Chiari, & Almeida, 2021; Yerushalmy & Shternberg, 2020). Johnson et al. (2023) also introduced ‘techtivities’ as boundary objects, and this supports faculty integration of digital tools.

**Discourse-Rich Pedagogies**- Discourse-rich pedagogies, utilizing commognitive theory, concludes that structured mathematical communication promotes algebraic reasoning (Sfard, 2020; Weber, Mejía-Ramos, & Alcock, 2022). Chamberlain (2023) adapted the argumentative knowledge construction framework to engage in a serial discussions. Viirman (2021) modeled university mathematics lecturing as mathematical discourse.

**Contextual Problem-Solving**- The report of Gravemeijer and Doorman (2020) revealed that Contextual problem-solving is rooted in RME and also supports progressive formalization. Another report, Hoyles and Noss, (2020) also supports the report.

#### Methodology

**Research Design:** This paper employed an integrative literature review relying on Torraco (2016), which is appropriate for synthesizing theoretical and empirical literature across diverse frameworks.

#### Search Strategy and Source Identification

Searches were conducted across ERIC, JSTOR, Scopus, Web of Science, and Google Scholar in November 2024. Search strings included combinations of: (algebraic thinking or algebraic reasoning) and (undergraduate or tertiary) and (mathematics majors or abstract algebra or structural reasoning). Additional terms are: representational fluency, proof comprehension, inquiry-based learning, and digital technology. Reference lists were reviewed for additional sources.

#### Inclusion and Exclusion Criteria

Sources included if they met all criteria: (1) published between 2015 and 2025 (foundational works from earlier periods included for theoretical coherence); (2) peer-reviewed journals, edited volumes, or recognized academic

publishers; (3) focused on algebraic thinking or algebra pedagogy at undergraduate level, or offered applicable theoretical frameworks; (4) addressed at least one of: conceptualization, theoretical frameworks, pedagogy, representational fluency, proof, technology; (5) English language. Excluded: K-12 focus without undergraduate relevance, or lacking scholarly rigor.

**Data Extraction and Analysis**

Thirty-five sources out of the available sources met inclusion criteria. For each, author(s), year, theoretical framework, objectives, methodology, key findings, and implications were extracted. Thematic analysis according to (Braun & Clarke, 2006) proceeded through familiarization, initial coding, theme development, review, and final definition. In this study, a hybrid inductive-deductive approach was used: initial codes were generated inductively from source texts, then, they were organized deductively against existing theoretical constructs (e.g., generalization, structural reasoning). Two reviewers independently coded 20% of sources; inter-rater reliability was 0.87 (Cohen’s  $\kappa$ ). Disagreements were resolved through discussion to reach consensus.

**Trustworthiness and Rigour**

To enhance trustworthiness, the search strategy was documented in full to support replication. Inclusion and exclusion criteria were applied uniformly across all 30 sources. Thematic analysis proceeded iteratively, with explicit attention to disconfirming evidence (negative case analysis) to avoid confirmation bias. Finally, the resulting four-dimensional framework was systematically checked against the entire corpus to ensure comprehensive representation of the literature

**Results**

**Dimensions of Algebraic Thinking for Undergraduate Mathematics Majors**

Thematic analysis identified four interconnected dimensions. Table 1 summarizes them.

**Table 1: Dimensions of Algebraic Thinking for Undergraduate Mathematics Majors**

Dimension	Definition	Key Indicators	Informing Frameworks	Key Sources
Generalization and Abstraction	Identifying, expressing, and reasoning about patterns and invariances across mathematical situations, including formulating abstract definitions and general proofs	Recognizing patterns across cases; formulating definitions; constructing general proofs; moving from instances to abstract structures	APOS Theory; Cultural-Semiotic Theory	Dubinsky & McDonald (2017); Mason (2018); Tall (2019); Radford (2018, 2022, 2024); Kieran (2021)
Structural Reasoning	Perceiving, analyzing, and reasoning about underlying structures of algebraic systems, attending to properties, relationships, and organizing principles	Identifying structural similarities (e.g., isomorphisms); reasoning about operations via defining properties; exploiting structure in problem-solving	Commognitive Theory; APOS Theory	Weber et al. (2022); Stacey & Chick (2021); Kieran (2021); Sfard (2020); Lee & Park (2024)
Representational Fluency	Creating, interpreting, translating between,	Coordinating symbolic, graphical, tabular,	RME; Cultural-Semiotic Theory	Janvier, C. (1987/2019); Yerushalmy &

Dimension	Definition	Key Indicators	Informing Frameworks	Key Sources
Proof-Based Algebraic Argumentation	and critically evaluating multiple representations of algebraic objects and relationships	and diagrammatic representations; translating between representations; using digital representations	Commognitive Theory; APOS Theory	Shternberg (2020); Borba et al. (2021); Radford (2022); Lee & Park (2024)
	Formulating precise mathematical statements and constructing, comprehending, and evaluating logically valid arguments using algebraic structures and properties	Writing epsilon-delta proofs; proving group properties; identifying logical gaps; using definitions to construct proofs; evaluating others' proofs		Mejía-Ramos et al. (2021); Cooley et al. (2024); Sfard (2020); Weber et al. (2022)

### Generalization and Abstraction

This foundational dimension involves identifying and expressing patterns, regularities, and invariances across mathematical situations (Mason, 2018; Kieran, 2021). At the undergraduate level, it includes formulating abstract definitions, putting together proofs that are true of whole classes of objects, and finding structure in common to different systems. According to Tall (2019), this is an axiomatic definition of objects into a formal world. Radford (2018) adds that generalization is a semiotically and culturally mediated process.

### Structural Reasoning

Structural reasoning shifts students from a procedural orientation toward a structural one: from attending to what operations produce symbolically to attending to what properties make those operations meaningful (Weber et al., 2022). It encompasses recognizing and exploiting similarities across algebraic contexts, reasoning about operations via defining properties, and deploying structural knowledge in proof construction and problem-solving (Stacey & Chick, 2021). Kieran (2021) focuses on the fact that algebraic thinking consists of three major activities: Generational (building expressions/equations), Transformational (rule-based manipulation), and Global/Meta-level (modelling and problem-solving).

### Representational Fluency

Representational fluency links concrete and abstract aspects of algebraic thought by allowing students to construct, read, and translate between more than two representations of algebraic objects and relationships. Janvier (1987/2019) suggested that the ability to flexibly move between symbolic, graphical, tabular, and verbal representations was a sign of strong mathematical knowledge, and an aid to discovery. In undergraduate algebra, representational fluency is especially important when the students learn about abstract constructions (e.g., groups, vector spaces) that have multiple isomorphic representations. The recent empirical studies have recorded representational fluency difficulties even among the tertiary students. Lee and Park (2024) evaluated the representational fluency of 34 students of Korean universities regarding the derivative concept in terms of items that need translations of an algebraic, graphical, numerical, and verbal presentation. Whereas students demonstrated high performance in certain items, they failed in verbal representation of the derivative meaning and application of the derivative concept to functions that were not represented in an algebraic way. Importantly, there were not many statistically significant correlations among translation skills that ought to be interrelated, indicating disjointed representational comprehension. This finding implies that representational fluency is not automatically acquired through symbolic manipulation and must be explicitly taught, even at the tertiary level. This result has direct implications on teaching algebra at the undergraduate level: students with the ability to manipulate symbolic expressions do not necessarily have the flexibility of reasoning across representations, which is a fundamental aspect of algebraic thinking at the tertiary level.

### Proof-Based Algebraic Argumentation

Undergraduate mathematics is characterized by the building up and understanding of proofs, to the point that it is a completely different subject compared to secondary school. In the sense of algebraic thought, proof-based algebraic argumentation is the skill to write down accurate statements about algebraic objects (e.g., groups, rings, fields, vector spaces), to construct, understand, and analyze arguments that are logically sound using algebraic properties and relationships. Significant evidence is writing epsilon-delta proofs in analytic situations that are algebraic, establishing group properties like associativity or closure, detecting logical gaps in given proofs, constructing original proofs by definition, and analyzing the validity of other algebraic arguments. Mejía-Ramos et al. (2021) in an empirical study revealed that the ability of students to understand and formulate proofs is closely related to their algebraic reasoning. A significant more recent input is Cooley et al. (2024), with a report of seven years of collaboration between research mathematicians and mathematics educators. Based on a model of understanding proof, they applied a three-year cycle of planning, implementing and assessing student responses at two universities. The resulting Promoting Reasoning in Undergraduate Mathematics (PRIUM) Qualitative Framework of Proof Comprehension offers a validated instrument used to measure how students understand proofs, instead of the definition-theorem-proof format that promotes memorization instead of syllogistic reasoning. This framework can be directly applied in evaluating the evidence-based algebraic argumentation in undergraduate algebra courses.

### Illustrative Vignettes

The following vignettes are illustrative and hypothetical, grounded in typical undergraduate algebra tasks described in the literature.

**Structural Reasoning:** An introductory group theory course provides students with a question whether the set of even integers under addition is a group. A student with structural reasoning can tell which properties of integer addition are preserved, closure (sum of evens is even), identity (0), inverses ( $-2k$ ), and associativity, without having to prove each axiom individually. Conversely, a student who uses procedural reproduction may also seek to enumerate all the elements or memorize a template.

**Representational Fluency:** When learning about linear transformations, a student with representational fluency moves swiftly and easily switches among the algebraic expression (matrix multiplication), the geometric expression (transformation of the plane: rotation, reflection, scaling) and the description (rotation by 90 degrees counterclockwise). A less fluent learner can algebraically determine a solution but fails to draw a diagram of what happens to a unit square or explain it in words.

### Proof-Based Algebraic Argumentation:

In an abstract algebra exam, students may be asked to prove that if  $G$  is a group and  $b^2 = e$  for all  $b \in G$  (where  $e$  is the identity element), then  $G$  is abelian. A student demonstrating strong proof-based algebraic argumentation constructs a logically sequenced justification: For any  $x, y \in G$ , we have  $(xy)^2 = e$  by the given property. Expanding yields  $xyxy = e$ . Left-multiplying by  $x$  and right-multiplying by  $y$  gives  $(xx)yx(yy) = xey$ . Since  $xx = x^2 = e$  and  $yy = y^2 = e$ , this simplifies to  $eyx = xy$ , and thus  $yx = xy$ . The student explicitly references the group axioms (closure, associativity, identity, and inverses) and the given involution property at each step. This example illustrates how proof-based argumentation integrates algebraic structure, symbolic manipulation, and logical deduction—three dimensions that distinguish undergraduate algebraic thinking from procedural school algebra.

### Pedagogical Approaches

Table 2 presents four major pedagogical approaches supported by the literature.

**Table 2: Pedagogical Approaches for Fostering Algebraic Thinking**

Pedagogical Approach	Key Features	Primary Dimensions Addressed	Supporting Evidence
Inquiry-Based Learning (IBL)	Student-centered exploration; conjecture and justification; collaborative problem-solving; instructor as facilitator	Generalization and Abstraction; Structural Reasoning; Proof-Based Argumentation	Laursen & Rasmussen (2019); Rasmussen & Wawro (2021); Zavala (2023); Andreasen & Misfeldt (2024)
Digital Technology Integration	Computer algebra systems; dynamic visualizations; interactive environments; blended learning	Representational Fluency; Structural Reasoning; Generalization	Borba et al., (2021); Yerushalmy & Shternberg (2020); Johnson et al., (2023)
Discourse-Rich Pedagogy	Mathematical writing; collaborative discussion; proof communication; reflective commentary	Proof-Based Argumentation; Structural Reasoning; Representational Fluency	Sfard (2020); Weber, Mejía-Ramos, & Alcock (2022); Chamberlain (2023); Viirman (2021)
Contextual Problem-Solving	Context problems as starting points; progressive mathematization; guided reinvention	Generalization and Abstraction; Representational Fluency	Gravemeijer & Doorman (2020); Hoyles & Noss (2020); Melaibari & Ismail (2023)

### Challenges and Tensions

Table 3 summarizes key challenges.

**Table 3: Challenges and Tensions**

Challenge	Description	Relevant Sources
Transition from school to university algebra	Sharp discontinuity between procedural school algebra and proof-based university algebra	Tall (2019); Weber et al. (2022); Kieran (2021)
Instructor preparation and pedagogical content knowledge	Deep content knowledge but limited pedagogical training; reform approaches difficult to implement	Nardi & Ryve (2023); Anthony & Hunter (2024)
Equity and access	Algebraic thinking shaped by cultural and linguistic contexts; fragmented representational understanding may disadvantage underrepresented students	Radford (2022); Kaput (2008/2019); UNESCO (2021)
Technology integration	Effective use requires deliberate pedagogical design; adoption without design reinforces procedural approaches	Borba et al. (2021); Yerushalmy & Shternberg (2020); Johnson et al. (2023)
Balancing context and rigor	Real-world contexts must avoid oversimplification or loss of mathematical generality	Hoyles & Noss (2020); Gravemeijer & Doorman (2020)

Challenge	Description	Relevant Sources
Assessment alignment	Timed examinations emphasizing procedural tasks do not capture full range of dimensions	Mejía-Ramos et al. (2021); Cooley et al. (2024)

### Integrated Conceptual Framework

Table 4 presents the proposed integrated framework.

**Table 4: Integrated Conceptual Framework**

Layer	Components	Interconnections
Theoretical Foundations	Cultural-Semiotic Theory; APOS Theory; Commognitive Theory; Realistic Mathematics Education	Provide epistemological, cognitive, discursive, and cultural grounding. Generalization identifies patterns (e.g., recognizing that $Z_n$ and symmetries of a regular n-gon both form groups); structural reasoning explains why (both satisfy group axioms); representational fluency expresses this via Cayley tables, cycle graphs, or permutation notation; proof-based argumentation formally establishes isomorphism.
Dimensions of Algebraic Thinking	Generalization and Abstraction; Structural Reasoning; Representational Fluency; Proof-Based Algebraic Argumentation	Generalization identifies patterns; structural reasoning explains why; representational fluency expresses them; proof-based argumentation establishes validity. Each dimension enables the others.
Pedagogical Approaches	IBL; Digital Technology Integration; Rich Discourse-Pedagogy; Contextual Problem-Solving	Each addresses multiple dimensions; effective instruction integrates approaches
Contextual Factors	Institutional context; instructor preparation; student diversity; assessment practices; policy frameworks	Shape implementation conditions; require attention to equity, access, alignment

### Discussion

This paper proposed a conceptualization of algebraic thinking for undergraduate mathematics majors. The four intersecting and complementary dimensions are: generalization and abstraction, structural reasoning, representational fluency, and proof-based algebraic argumentation. These dimensions have been developed as a result of the thematic synthesis and can be concisely summarized in Table 1. The four theoretical frameworks discussed in this paper present complementary insights on these dimensions. The historically and semiotically mediated nature of algebraic cognition is foregrounded by cultural-semiotic theory (Radford, 2018, 2022, 2024). The procedural-manipulation-structural-understanding developmental pathway is charted by APOS theory (Dubinsky & McDonald, 2017). The constitutive role of discourse is identified as a part of the development of algebraic thinking according to commognitive theory (Sfard, 2020). The emphasis of Realistic Mathematics Education (Gravemeijer & Doorman, 2020) is on situationalized grounding and formalization in a gradual manner. No single framework can explain the entire concept of algebraic thinking at the undergraduate level. Their combination, as suggested in this paper, offers a conceptualization that is sufficient for the needs of tertiary algebra

teaching and study. Nevertheless, the proposed framework remains theoretical. The relationships among the four dimensions, for example, whether they develop sequentially or in parallel are not yet empirically established.

## Conclusion

Algebraic thinking is a multidimensional, culturally mediated, and developmentally complex competency at the heart of undergraduate mathematics education. This paper has responded to the undertheorized nature of algebraic thinking for mathematics majors by proposing an integrated conceptual framework organized around four interconnected dimensions: generalization and abstraction, structural reasoning, representational fluency, and proof-based algebraic argumentation. Grounded in a synthesis of cultural-semiotic theory, APOS theory, commognitive theory, and Realistic Mathematics Education, the framework acknowledges that no single theoretical perspective adequately captures the full scope of algebraic thinking at the tertiary level. Instead, their integration offers a more robust and nuanced understanding. The findings carry several implications for practice. Curriculum design should deliberately address all four dimensions in an integrated manner, moving beyond the sequential or siloed treatment that often characterizes undergraduate algebra instruction. Pedagogical approaches such as inquiry-based learning, digital technology integration, discourse-rich pedagogy, and contextual problem-solving each target multiple dimensions and, when combined, can foster the kind of structural reasoning and proof-based argumentation expected of mathematics majors. Assessment practices must also evolve to capture the full range of algebraic thinking, including representational fluency and proof comprehension, rather than privileging procedural fluency alone.

It is important to acknowledge the following limitations. First, the conceptualization is theoretical and has not been empirically tested to be a comprehensive model of measuring or developing algebraic thinking. Second, integration of frameworks is necessarily selective and interpretive; alternative frameworks (e.g., instrumental genesis, variation theory) might contribute some supplementary information. Third, the review did not consider non-English sources, which may have overlooked possibly relevant research from non-Anglophone contexts. A subsequent systematic review incorporating non-English databases (e.g., CNKI, SciELO) is recommended. Fourth, pedagogical implications are presented in general terms that could use context-specific details for particular courses and student populations. Several directions follow from the present study. Empirical validation was noted as the most pressing need: there is the need to test the four-dimensional model, using design-based research, case studies and large-scale assessments. Furthermore, development of assessment must be directed to the designing and validation of the tools that will capture structural reasoning, representation fluency, and proof based argumentation successfully. Also, longitudinal studies of development of algebraic thinking of the undergraduate mathematics majors, would be more adequate to explain the developmental patterns. Comparative institutional studies and cross-cultural research could also reveal the influence of contextual factors on development. Specific pedagogical strategies should be compared with the dimensions identified in intervention studies, with specific rigorous experimental and quasi-experimental designs. Studies on integration with other mathematical competencies of the mathematics majors (geometric reasoning, probabilistic thinking, and computational thinking) would guide more consistent curriculum design. Anthony and Hunter (2024) in a study suggests further research in longitudinal views and shifts in emphases across reviews. When the cognitive, discursive, representational, and argumentative dimensions of algebraic thinking are made explicit and when instructors know precisely what they are trying to develop and why, teaching shifts from covering content to building mathematical thinkers. That shift is what undergraduate mathematics education, at its best, demands.

## Suggestions

### Implications for Practice

The present study has implications for the curriculum designer. Curriculum should clearly focus on all four dimensions in an integrated form, rather than sequential manner. For instance, a task on group axioms could ask students to:

- a. Generate examples from a given text (generalization)
- b. Compare the group structure of integers mod  $n$  and symmetries of a square (structural reasoning)
- c. Represent the groups using Cayley tables and cycle graphs (representational fluency)
- d. Prove or establish whether a given set with an operation forms a group or not (proof-based argumentation).

At tertiary level, task design should no longer be regarded as routine procedural exercises but as open-ended tasks requiring justification, conjecture, communication and generalization. The integration of technology must be both purposeful and pedagogically informed at this level. Though, digital tools have the capacity to develop

representational fluency in students, but this potential requires intentional design choices about what students do with those tools and why (Borba et al., 2021; Johnson et al., 2023). Classroom discourse should create structured opportunities for mathematical writing, collaborative discussion, and reflective commentary (Chamberlain, 2023). Instructor development should extend to pedagogical content knowledge in algebra; faculty need access to the research literature and reform-oriented pedagogical models (Anthony & Hunter, 2024). Assessment practices should reflect the full scope of algebraic thinking. Instruments that capture only procedural fluency tend to leave structural reasoning, representational flexibility, and proof comprehension unmeasured (Cooley et al., 2024)

### Implication for Equity and Access

Kaput (2008/2019) reported that algebra has been a gatekeeping mechanism for ages. Radford (2022) and UNESCO (2021) do not only consider the cultural, but also the linguistic nature of algebraic thinking and suggest that inclusive pedagogy should take into account various semiotic resources. The theories advocated in this paper, in particular, IBL and discourse-based pedagogy, could enhance a more inclusive learning environment as they justify multiple approaches to problem-solving and promote the joint construction of knowledge. In practice, this means designing tasks that accept multiple semiotic resources (words, gestures, diagrams) and avoid privileging only symbolic manipulation, which can disadvantage students from non-dominant linguistic backgrounds.

### References

- Andreasen, K., & Misfeldt, M. (2024). Inquiry-based linear algebra teaching and learning in a flipped classroom framework. *Teaching Mathematics and its Applications*, 43(1), 45–62.
- Anthony, G., & Hunter, J. (2024). The teaching and learning of tertiary mathematics. In J. Anderson & K. Makar (Eds.), *Research in mathematics education in Australasia 2020–2023* (pp. 187–211). Springer.
- Bilondi, M. T., & Radmehr, F. (2023). Students' mathematical thinking of the tree concept: An integration of APOS with Tall's three worlds of mathematics. *Research in Mathematics Education*, 25(3), 367–388.
- Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2015). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3–5*. National Council of Teachers of Mathematics.
- Borba, M. C., Chiari, A., & Almeida, H. R. F. L. (2021). Interactions between digital technology and algebraic thinking: A blended approach. *Educational Studies in Mathematics*, 107(1), 67–85.
- Borji, V., & Martínez-Planell, R. (2024). Students' understanding of Riemann sums and double integrals: The case of task design in APOS theory. *International Journal of Research in Undergraduate Mathematics Education*. Advance online publication.
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77–101.
- Carraher, D., & Schliemann, A. (2018). Early algebra and algebraic reasoning: A current perspective. *ZDM – Mathematics Education*, 51(1), 1–13.
- Chamberlain, D. (2023). Adapting the argumentative knowledge construction framework to asynchronous mathematical discussions. *International Journal of Research in Undergraduate Mathematics Education*, 9(2), 345–372.
- Cooley, L., Dorfmeister, J., Miller, V., Duncan, B., Littmann, F., Martin, W., Vidakovic, D., & Yao, Y. (2024). The PRIUM qualitative framework for assessment of proof comprehension: A result of collaboration among mathematicians and mathematics educators. *ZDM – Mathematics Education*, 56, 1553–1566.
- Dubinsky, E., & McDonald, M. A. (2017). APOS theory: A framework for research and curriculum development in mathematics education. *Constructivist Foundations*, 12(2), 58–73.
- Gravemeijer, K., & Doorman, M. (2020). Context problems in realistic mathematics education: A reformulation of algebra teaching. *ZDM – Mathematics Education*, 52(6), 1127–1138.
- Hoyles, C., & Noss, R. (2020). Translating between abstract and concrete: Real-world contexts in undergraduate algebra. *Mathematics Education Research Journal*, 32(3), 307–324.
- Jamil, N. B., Rosli, R. B., Mahmud, M. S. B., & Hasim, S. B. M. (2025). Transformative teaching strategies for algebraic thinking: A systematic review of cognitive, pedagogical, and curricular advances. *Eurasia Journal of Mathematics, Science and Technology Education*, 21(10), Article em2721.
- Janvier, C. (2019). Representation and understanding: The notion of function as an example. *International Journal of Mathematical Education in Science and Technology*, 50(7), 1065–1080. (Original work published 1987)
- Johnson, H. L., Olson, G., Tsinnajinnie, B., & Bechtold, L. (2023). Techivities as boundary objects for promoting instructional transformation in early undergraduate mathematics courses. *Digital Experiences in Mathematics Education*, 9(1), 1–25.

- Kaput, J. (2019). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. *Research in Mathematics Education*, 21(2), 102–118. (Reprinted from 2008)
- Karavi, T., Mali, A., & Avraamidou, C. (2022). Commognition as an approach to studying proof teaching in university mathematics lectures. *International Journal of Mathematical Education in Science and Technology*, 53(8), 2015–2034.
- Kieran, C. (2018). Evolving research on algebra learning. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developments in mathematics education research* (pp. 25–40). Springer.
- Kieran, C. (2021). Evolving perspectives on algebraic thinking in school mathematics. *For the Learning of Mathematics*, 41(1), 3–9.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.
- Kontorovich, I. (2023). When learning stumbles upon identity and affect: A loaded student–student collaboration in linear algebra. *International Journal of Mathematical Education in Science and Technology*, 54(8), 1461–1477.
- Laursen, S., & Rasmussen, C. (2019). Inquiry-based learning in undergraduate mathematics. *Journal of Mathematical Behavior*, 55, 100703.
- Lee, S., & Park, J. (2024). An analysis of university students' representational fluency about the derivative concept. *Journal of Educational Research in Mathematics*, 34(2), 215–238.
- Mason, J. (2018). Generalization and algebra: Exploiting children's powers. *Quadrante*, 27(2), 75–98.
- Mejía-Ramos, J. P., Lew, K., & Weber, K. (2021). Students' proof comprehension and algebraic reasoning in undergraduate mathematics. *Journal for Research in Mathematics Education*, 52(5), 487–515.
- Melaibari, O. M., & Ismail, N. (2023). The effect of realistic mathematics education on undergraduate freshmen students' mathematical competencies. *Applied Mathematics and Information Sciences*, 17(1), 55–66.
- Molina, V. M., Regaña, A. J. G., Barragán, R. T., & Izquierdo, J. M. G. (2023). Differences between how undergraduate students define geometric solids and what their lecturers expect from them through the lens of the theory of commognition. *Eurasia Journal of Mathematics, Science and Technology Education*, 19(12), 1917–1926.
- Mulligan, J., & Woolcott, G. (2024). Mapping the transition from arithmetic to algebra: A bibliometric and structural review (2003–2023). *Frontiers in Education*, 9, 1356247.
- Nardi, E., & Ryve, A. (2023). University mathematics teaching: Shifts in pedagogy in the 21st century. *International Journal of Research in Undergraduate Mathematics Education*, 9(2), 145–168.
- Radford, L. (2018). The emergence of symbolic algebraic thinking in elementary school: A cultural-semiotic perspective. *Educational Studies in Mathematics*, 98(1), 61–79.
- Radford, L. (2022). The embodied, cultural, and historical dimensions of algebraic thinking. *Educational Studies in Mathematics*, 109(3), 401–419.
- Radford, L. (2024). The cultural-historical dimension of algebraic thinking: A semiotic approach. In *Handbook of the cultural foundations of learning* (pp. 245–262). Routledge.
- Rasmussen, C., & Wawro, M. (2021). Reform-oriented pedagogy in undergraduate mathematics. *Notices of the American Mathematical Society*, 68(5), 732–743.
- Reyes-Rodríguez, A., & Santos-Trigo, M. (2024). Diagnostic study of mathematical reasoning in novice university students. *International Electronic Journal of Mathematics Education*, 19(3), em0778.
- Sfard, A. (2020). *Thinking as communicating: Human development, the growth of discourses, and mathematizing* (2nd ed.). Springer.
- Stacey, K., & Chick, H. (2021). Solving the problem with algebra problems: Rethinking algebraic thinking. *Mathematics Education Research Journal*, 33(4), 529–545.
- Students' use of resources in a challenge-based learning context. (2021). *International Journal of Research in Undergraduate Mathematics Education*, 7(3), 456–478.
- Tall, D. (2019). Abstraction and the transition to advanced mathematical thinking. *Mathematics Education Research Journal*, 31(2), 89–103.
- Torraco, R. J. (2016). Writing integrative literature reviews: Guidelines and examples. *Human Resource Development Review*, 15(4), 404–429.
- UNESCO. (2021). *Reimagining our futures together: A new social contract for education*. UNESCO Publishing.
- Vargas, C. G., & Okaç, A. (2024). Mental constructions for the learning of the concept of vector space. *ZDM – Mathematics Education*, 56, 1417–1431.
- Viirman, O. (2021). University mathematics lecturing as modelling mathematical discourse. *International Journal of Mathematical Education in Science and Technology*, 52(6), 881–901.

- Weber, K., Lew, K., & Mejía-Ramos, J. P. (2022). Rethinking algebra instruction at the tertiary level: From procedures to structures. *Educational Studies in Mathematics*, 111(2), 237–258.
- Weber, K., Mejía-Ramos, J. P., & Alcock, L. (2022). Written explanations in undergraduate mathematics: Promoting algebraic reasoning through communication. *Journal for Research in Mathematics Education*, 53(2), 123–145.
- Yerushalmy, M., & Shternberg, R. (2020). Digital technologies and the teaching of algebra: Linking representations. *Digital Experiences in Mathematics Education*, 6(2), 197–218.
- Zavala, M. (2023). Examining the effect of inquiry-based learning versus traditional lecture-based learning on students' achievement in college algebra. *International Electronic Journal of Mathematics Education*, 18(1), Article em0724.