Faculty of Natural and Applied Sciences Journal of Mathematics and Science Education
Print ISSN: 2814-0885 e-ISSN: 2814-0931
www.fnasjournals.com
Volume 5; Issue 2; March 2024; Page No. 97-101.

A Study of Conditional Expectation on Filtered Fermionic Fock Space

***Ahmad, Z.T., & Tijjani, A.A.**
Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria

***Corresponding author email:** ahmadzulaihatj@gmail.com

Abstract
This paper presents a structural analysis of stopping time in noncommutative probability theory within the context of Fermionic Fock Space. In particular, the construction of conditional expectation and the introduction and analysis of stopping times on an antisymmetric Fock space are undertaken using an increasing full right continuous filtered antisymmetric Fock Space. The conditional expectation on the filtered Fock space is demonstrated to be completely positive.

Keywords: Fock Space, Fermionic Fock Space, Conditional Expectation, Stopping Time.

Introduction

The theory of rings of operators on Hilbert space began in the 1930s with a series of papers by von Neuman 1936, Murray and von Neuman 1937. They analyzed in great detail the structure of a family of algebras of operators which are nowadays referred to as *-algebras (B*-, C*-, W*-algebra). Among these *-algebras, W*- algebra (von Neumann) has the distinctive property of being rich in projection and closed in the weak operator topology. The study of stochastic processes like conditional expectation and stopping times is of great importance in the quantum probability theory, operator theory and quantum physics. For example, the quantum statistical description of the dilation of the dynamic of a quantum open system. The notion of classical conditional expectation in probability theory was first extended to the noncommutative probability theory by Umegaki(1954), he showed the existence of conditional expectation when the von Neumann Algebra is finite. Later Umegaki (1956), Tomiyama (1957), Umegaki(1958), Umegaki Nakamura (1961) and Takesaki (1972) studied conditional expectation in this direction. Following the work of Evans (1979), Barnett, Streater and Wilde, (1983) constructed conditional expectation from a CAR C*-algebra to it CAR C*-subalgebra and showed it to be a composition map.

Hudson (1979) introduced Quantum stopping time as a projection-valued process, and Parthasarathy and Sinha (1987) extended the work of Hudson(1979) to the case of Boson Fock space. Barnett and Lyons (1986) introduced stopping time in antisymmetric Fock Space. Other researchers like Barnett and Thakrar (1988), Barnett et al. (1996), Attal and Coquio (2004) and Coquio (2006) further studied stopping times in Quantum Probability theory. Most recently, Kang (2015) defined quantum stopping time in the Interacting Fock Space over $L^2(\mathbb{R}^+)$ and developed a corresponding Quantum Stopping Time Stochastic Integral. In this research, we consider an increasing full right continuous filtration of Fermionic Fock space in constructing conditional expectation and defining quantum stopping time. This work tends to extend the work of Barnett and Lyons (1986) and that of Barnett et al. (1983) to the Antisymmetric Fock spaces.

2.0 Preliminaries

We give some basic definitions and theorems that are known in the literature, taken from Sakai (1971), Bing-Hen (1992) and Meyer (1993).

Definition 2.1: Let $B(H)$ be the set of all bounded linear operators acting on a complex separable Hilbert Space H. A C*- algebra U is a $*$-subalgebra of $B(H)$ which is uniformly closed and has the property, $\|x^*x\| = \|x\|^2$ for all $x \in U$. A von Neumann Algebra M is an unital *-subalgebra of $B(H)$ that is self-adjoint and closed with respect to the weak operator topology.
Let ψ be a linear functional on M, then ψ is said to be

Positive: If $\psi(a) \geq 0$ for any $\in M^{+}$,
Faithful: if $\psi \geq 0$ and $\psi(a) = 0$ implies $a = 0$, for some $a \in M^{+}$,
Normal: If $a_i \to a$ implies $(\sup a_i) = \sup \psi(a_i) = \psi(a)$,
State: if $\psi \geq 0$ and $\|\psi\| = 1$,
Trace: if $\psi \geq 0$ and $\psi(a^*a) = \psi(aa^*)$, for all $a \in M$,
Weight: if $\psi \geq 0$ and $\psi(\lambda a + b) = \lambda\psi(a) + \psi(b)$, for all $\lambda \geq 0, a, b \in M^{+}$,
where M^{+}= {a ∈ M: a ≥ 0}.

Theorem (GNS Construction): Let U be an unital C*- algebra and let $\psi : U \to \mathbb{R}$ be a state. Then there exists a triplet $(H_\psi, \pi_\psi \Omega_\psi)$ where

- $\pi_\psi : U \to B(H_\psi)$ is a *-representation of U, H_ψ being a Hilbert Space;
- $\Omega_\psi \in H_\psi$ and $H_\psi = \overline{\pi_\psi \psi(M) \Omega_\psi}$; and
- $\psi(a) = \langle \pi_\psi \psi(M) \Omega_\psi, \Omega_\psi \rangle$ for all $a \in M$.

Definition 2.2: A fock space is the direct sum of the tensor product of Hilbert space, that is, $F(H) := \bigoplus_{n\geq 0} H^n$, where $H^n = \mathbb{C}$ and $H^n = \bigotimes_{i\geq 1} H_i = H_1 \otimes H_2 \otimes H_3 \otimes \ldots \otimes H_n$.
A unit vector $\Omega \in F(H)$ is called the vacuum vector.

Definition 2.3: Given f ∈ H, we define the creation operator $a^*(f): H^n \to H^{n+1}\, by$

$$a^*(f) f_1 \otimes f_2 \otimes \ldots \otimes f_n = \sqrt{n+1}\, f \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_n$$

and the annihilation operator $a(f): H^n \to H^{n-1}$, $n \geq 1$ by

$$a(f) f_1 \otimes f_2 \otimes \ldots \otimes f_n = \sqrt{n}\, \langle f, f_1 \rangle f_2 \otimes \ldots \otimes f_n \text{, where } a(f) : H^0 \to 0 \in F(H)$$

The Fermionic creation and annihilation operators satisfy the Canonical Anticommutation Relation (CAR)

$$[a(f), a^*(g)] = \langle f, g \rangle \mathbf{1}_{F(H)} ,$$

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0 .$$

Where $[a(f), a^*(g)] = a(f)\, a^*(g) + a^*(g) a(f)$

Definition 2.4: A CAR Algebra U over H is defined to be an unital C*- Algebra generated by elements $a(f)$ and $a^*(f)$, $f \in H$ which satisfy the canonical anticommutation relation.

Definition 2.5(Evans & Lewis, 1977): Let T be a linear map from a $*$-Algebra A to $*$-Algebra B, let T_n denote the product mapping $T \otimes 1_n$ from $\mathbb{M}_n(A)$ into $\mathbb{M}_n(B)$, where 1_n denotes the identity mapping on $\mathbb{M}_n(\mathbb{C})$, that is $T_n : [a_{ij}] \longmapsto [T(a_{ij})]$. T is said to be positive if it maps positive elements of A to positive elements of B and T is completely positive if every T_n is a positive map.

Definition 2.6 (Barnett et al., 1983): Let U be a CAR algebra over H and V be its CAR subalgebra. A conditional expectation is a bounded linear map $E_t: U \to V$ satisfying:

E1 $E_t(x) = x$ for all $x \in V$,
E2 $E_t(xyz) = xE_t(y)z$, for $x, z \in V$, $y \in$ U ,
E3 E_t: $\mathrm{U}^+ \longrightarrow V^+$,
E4 $E_t(x)^* E_t(x) \leq E_t(x^*\mathrm{x})$, x ∈ U , $\|E_t\| = 1$.

Remark: Conditional expectation E_t is a positive contractive bounded linear map which leaves the elements of the subalgebra fixed and whose norm is one.

Definition 2.7(Barnett & Thakrar, 1990): Let M be a von Neumann Algebra acting on H, a stopping time τ, is an increasing family $(q_t)_{t\in\mathbb{R}^+}$ of projections in (M_t).

$$\tau : \mathbb{R}^+ \to (M_t)_{proj} \text{ such that: } \tau(t) = (q_t) \in (M_t)_{proj},\ for\ t \in (0, \infty)$$

$$\tau(0) = 0, \tau(\infty) = I.$$

3.0 Results

In this section, we construct conditional expectations in filtered antisymmetric fock space and analyze stopping time in the aforementioned space. We begin with the following definition.

Definition 3.1: Let U be a CAR algebra acting on separable Hilbert Space H and U_t be a CAR subalgebra of U acting on separable Hilbert Space $H_t \subset H$. Then a filtration is a family $\{U_t : t \in \mathbb{R}^+\}$ satisfying:

i. if $t, s \in \mathbb{R}^+$ and $0 \le t \le s$ then $U_t \subseteq U_s$ (Increasing),
ii. $\overline{\cup_{t\geq 0} U_t}^{\|.\|} = U$ (Full),
iii. $\cap_{s>t} U_s = U_t$ (Right continuous)

and the family $\{H_t : t \in \mathbb{R}^+\}$ satisfies the following conditions:

i. if $t, s \in \mathbb{R}^+$ and $0 \le t \le s$ then $H_t \subseteq H_s$,
ii. $\overline{\cup_{t\geq 0} H_t}^{\|.\|} = H$,
iii. $\cap_{s>t} H_s = H_t$.

Lemma 3.1: The identity mapping $I\colon \mathrm{U}_t \longrightarrow \mathrm{U}_t$ is completely positive

Proof
The identity map I is positive since it maps positive operators in U_t to positive operators in U_t. To show that I is completely positive, we let $\mathbb{M}_n(\mathrm{U}_t) = \mathrm{U}_t \otimes \mathbb{M}_n$ be an $n \times n$ matrix with entries from U_t. Then,

$$I_n = I \otimes 1_n\colon \quad \mathbb{M}_n(\mathrm{U}_t) \longrightarrow \mathbb{M}_n(\mathrm{U}_t)$$
$$\left[a(f)_{ij}\right] \longmapsto \left[I(a(f)_{ij})\right].$$

Since $I(a(f)) \geq 0$, then $I(a(f)_{ij})$ is also positive for all ≥ 1 . Therefore I is completely positive.
We now construct a mapping $E_t\colon U \to U_t$ with range in the filtered CAR algebra.
Let $\{U_t : t \in \mathbb{R}^+\}$ be a Filtered CAR C*- Algebra over H, then U_t is an associative, unital *-Algebra generated by $a(f)$ and $a^*(f)$, for $f \in H$ with a C* - norm $\|a(f)\| = \|a^*(f)\| = \|f\|$, for $f \in H$. Suppose φ is n invariant quasi-free state on U_t, since φ is faithful, without loss of generality, the triplet (H_t , π, Ω) is the associated cyclic representation of U_t . Let β be an automorphism of U_t defined as $\left(a^*(f)\right) = a^*(f)$, $f \in H$. Then φ is invariant under β and so in the $*$-representation π, β is a unitary operator, denoted by ϒ, with $\Upsilon\Omega = \Omega$.
Suppose $H^n = H_1 \oplus H_2 \oplus \dots \oplus H_n$ and U_t be a filtration of U, then for any $f = f_1 \oplus f_2 \oplus \dots \oplus f_n$.
We now define $\pi : U \to U_t \otimes B(H)$ as $\pi\left(a(f_1 \oplus f_2)\right) = a(f_1) \otimes \Upsilon + 1 \otimes \pi\left(a(f_2)\right)$.
Since $a^*(f) = a(f)^*$, set $\pi(a^*(f)) = \pi(a(f))^*$, then for $f \in H$, $\pi(a(f))$ and $\pi(a^*(f))$ satisfies the canonical anticommutation relation. And so, it follows that π extends to an injective $*$-homomorphism of U into $U_t \otimes B(H^n)$ which we also denote by π. By Lemma 3.1 above I is completely positive and the map from $\mathfrak{S}\colon B(H) \longrightarrow \mathbb{C}$ is completely positive and so the map $I \otimes \mathfrak{S}$ extends to an algebraic tensor product $U_t \otimes B(H^n)$ to define a completely positive map

$$\theta : U_t \otimes B(H^n) \to U_t$$
$$\pi(a(f)) \longmapsto \theta(\pi(a(f))) \in U_t$$

that is $\theta o\pi = E_t : U \to U_t$ is the composition of π with θ.

Theorem 1: The map $E_t = \theta o \pi : U \to U_t$ is a conditional expectation in the filtered CAR C^*-algebra.

Proof
Since both ϑ and π are bounded and positive maps, the E_t is also positive and bounded which gives (E3).
Let $x = a^*(f_1)a^*(f_2) \dots a^*(f_n)a(g_1)a(g_2) \dots a(g_n)$ for $f_1, f_2, \dots f_n, g_1, \dots g_n \in H_t$, from the fact that $\Upsilon\Omega = \Omega$, $E_t(x) = x$, but such x generate U_t, and so by linearity and continuity, we have $E_t(x) = x$ for $x \in \mathrm{U}_t$ proving (E1).
Now for $x, z \in \mathrm{U}_t$ and $y \in \mathrm{U}$, $E_t(xyz) = E_t(x)E_t(y)E_t(z) = xE_t(y)z$ which gives (E2)
Similarly, for $y \in U$, we have $\|E_t(y)\|^2 \le \|y\|^2 \Rightarrow \|E_t(y)\| \le \|y\|$

$$\Longrightarrow \|E_t\| \le 1.$$

Hence, E_t is structure-preserving and $\|E_t\| = 1$ proving (E4).

Theorem 2: The conditional expectation $E_t : U \to U_t$ is completely positive.

Proof
From Theorem 1 above, $E_t: U^+ \to {U_t}^+$ which implies $E_t \geq 0$.
Let $\mathbb{M}_n(U) = U \otimes \mathbb{M}_n$ and $\mathbb{M}_n(U_t) = U_t \otimes \mathbb{M}_n$.
Defining $E_t^n = E_t \otimes 1_n: \mathbb{M}_n(U) \longrightarrow \mathbb{M}_n(U_t)$ as $E_t^n([u_{ij}]) = [E_t^n(u_{ij})]$,
since $E_t \geq 0$, then $[E_t^n(u_{ij})] \geq 0$ which implies $\sum_n [E_t^n(u_{ij})] \geq 0$, that is, E_t^n is positive for every n and hence, E_t is completely positive.
Let $P_t: H = \overline{U\Omega}^{\|\cdot\|} \longrightarrow \overline{U_t\Omega}^{\|\cdot\|} = H_t$ be the extension of E_t onto H such that

$$P_t(u\Omega) = E_t(\mathrm{u})\Omega.$$

Then, P_t is an orthogonal projection on H.

Proposition 3.1: P_t lies in the commutant of U_t.

Proof. Let $v \in U_t$, u ∈ U, then

$$P_t(vu)\Omega = E_t(vu)\Omega = E_t(u)\mathrm{v}\Omega$$
$$= vE_t(u)\Omega = vP_t(u)\Omega \,.$$

Since $u\Omega$ is dense in H_t , the result follows.

Definition 3.2: Let $(U_t)_{Proj}$ denote the set of all projections in U_t . A stopping time, , is an increasing family $(q_t)_{t\in\mathbb{R}^+}$ of projections in $(U_t)_{Proj}$.

$$\tau : \overline{\mathbb{R}^+} \to (U_t)_{Proj}$$
$$t \mapsto \tau(t) = q_t$$

such that:
i $\tau(t) = (q_t) \in (U_t)_{Proj}$, for $t \in (0,\infty)$
ii $\tau(0) = 0$
iii $(\infty) = I$.

Definition 3.3: Let $\tau = (q_t)_{t\in\mathbb{R}^+}$ and $\sigma = (r_t)_{t\in\mathbb{R}^+}$ be two stopping times, then we define an order $\tau \leq \sigma$ if and only if $r_t \leq q_t$ for each t.
Let $\wp$ denote the set of all finite partitions of $[0,\infty]$. Then for $T \in \wp$ say $T = \{t_0,\dots,t_n\}$, we define an operator $P_{\tau(T)}$ on H as$P_{\tau(T)} = \sum_{i=1}^n (q_{t_i} - q_{t_i-1})P_{t_i} = \sum_{i=1}^n \Delta q_{t_i} P_{t_i}$.

Theorem 3: Let $\tau(t) = (q_t) \in (U_t)_{Proj}$ be a stopping time. Then

(i) $P_{\tau(T)}$ is an orthogonal projection,
(ii) if $T_1, T_2 \in \wp$ with T_2 a refinement of T_2 then $P_{\tau(T_2)} \leq P_{\tau(T_1)}$,
(iii) if $\sigma = (r_t)_{t\in\mathbb{R}^+}$ is another stopping time with $\tau \leq \sigma$ then $P_{\tau(T)} \leq P_{\sigma(T)}$ for $T \in \wp$.

Proof: See Theorem 2.3 of Barnett and Thakrar (1990).

Remark: From the above theorem, $P_{\tau(T)}$ is a decreasing family of projections, so we take the infimum and we have the following definition.

Definition 3.4: Let $\tau(t) = (q_t) \in (U_t)_{Proj}$ be a stopping time, we define the time projection at τ denoted by P_τ, as

$$P_\tau = \inf \sum_{i=1}^n \Delta q_{t_i} P_{t_i} = \inf P_{\tau(T)}$$

Theorem 4: For two stopping times τ, σ with $\tau \leq \sigma$, we have $P_\tau \leq P_\sigma$.

Proof: we just take the limit in theorem 3(iii).

4.0 Conclusion

In this study of noncommutative probability theory, conditional expectation and quantum stopping times were studied in antisymmetric Fock space. The basic difference between this work and the existing one (Barnett et al., 1993) is the use of Filtration of CAR Algebra. The study proved that the conditional expectation is a completely positive composition mapping, also using the filtered CAR Algebra the manuscript introduces stopping time as a family of projections. The concept of time projection is also defined in the filtered CAR Algebra setting.

References

Attal, S., & Coquio, A. (2004). Quatum stopping time and quasi-left continuity. *Elsevier*., 40, 497–512.

Barnett, C., Goldstein, S., & F.Wilde, I. (1996). Quatum stopping times and Doob-Meyer decomposition. *Operator theory*., 35, 85–106.

Barnett, C., & Lyons, T. (1986). Stopping non-commutative processes. *Math. Proc. Gamb. Phil. Soc*., 99, 151–161.

Barnett, C., Streater, R., & Wilde, I. F. (1983). Quasi-free quantum stochastic integrals for the CAR and CCR. *Journal of functional analysis*., 52, 19 – 47.

Barnett, C., & Thakrar, B. (1990). A non-commutative random stopping theorem. *Journal of Functional analysis*., 88(2), 342–350.

Bing-Hen, L. (1992). Introduction to Operator algebras, volume 1. *World Scientific*.

Coquio, A. (2006). The optional stopping theorem for quatum martingales. *Elsevier*., 238, 149–180.

Evans, D. E. (1979). Completely positive quasi-free maps on the CAR algebra. *Commun. Math. Phys*., 70, 53–68.

Evans, D. E., & Lewis, J. T. (1977). Dilations of irreversible evolutions in algebraic quantum theory, volume 24. *Comm. Dub. Inst. Adv. Studies*.

Hudson, R. L. (1979). The strong Markov property for canonical wiener processes. *Journal of Functional Analysis*, 34, 266–281.

Kang, Y. (2015). Quantum stopping times stochastic integral in the interacting fock space. *Journal of Mathematical Physics*, 56.

Meyer, P. A. (1993). Quantum Probability for Probabilists. *Springer-Verlag Berlin Heidelberg*., New York.

Murray, F. J., & von Neumann, J. (1937). On rings of operators II. *Trans. Math. Soc*., 41, 208–248.

Parthasarathy, K., & Sinha, K. B. (1987). Stop times in Fock space stochastic calculus. *Probab. Th. Rel. Field*., 75, 317–349.

Sakai, S. (1971). C*-algebras and W*-algebras. *Springer-Verlag Berlin Heidelberg*., New York.

Takesaki, M. (1972). Conditional expectation in von Neumann algebras. *J. Funct. Analy*., 9, 306–321.

Tomiyama, J. (1957). On the projection of norm one in w*-algebras. *Mathematical Institute, TShoku University*., 33, 608–612.

Umegaki, H. (1954). Conditional expectation in operator algebras. *Tohoku math*, 6:177–181.

Umegaki, H. (1956). Conditional expectation in an operator algebra, 11, *Tokohu Math*. J. 8, 86-100. 38.

von Neumann, J. (1936). Rings of operators. *Ann. Math*., 37, 116–229.