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A NONLINEAR SEPARABLE QUADRATIC OBJECTIVE FUNCTION WITH LINEAR CONSTRAINTS: A LEAST SQUARES APPROACH TO OPTIMISATION PROBLEMS

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Abstract
A least square technique of tackling optimization problems with nonlinear separable quadratic objective functions with linear constraints was developed by the study. Two problem sets were computed to confirm the efficiency of the developed approach. To confirm the authenticity of the solution, the Wolfram Mathematica programming software was employed to solve the examples illustrated in this study and the results obtained were in agreement as achieved using the least squares method. It was concluded that the algorithms of solving optimization problems with nonlinear separable quadratic objective function with linear constraints stated in this study were quite explicit, efficient and very easy to employ

Keywords: Constrained Programming Problem, Separable Quadratic, Symmetric Matrix, Maximum Likelihood

Introduction
Classical optimization theory develops the use of differential calculus to obtain the pen of maxima and minima known as points of extremes, for unconstrained and constrained functions (Wang et al., 2020). The study's objective is to employ the least squares technique for solving a Separable Quadratic Constrained Programming Problem with linear constraints. Hence, in a constrained nonlinear optimization problem, there is the presence of constraints, comprising the objective function and constraints (Al-Mumtazah & Surono, 2020). To Minimize/Maximize $f(X)$ for all the values of $X = (x_1, x_2, \dots, x_n)$ is what the problem seems to address, subject to the constraints. The condition that must be met in order for a solution to be optimal is $\frac{\partial f}{\partial x_j} = 0$ for $j = 1,2,3, \dots, n$ at $x = X^*$.

Where ∂f is differentiable (Mahajan & Gupta, 2019). The condition is also a sufficient condition for maximization if the function $f(X)$ is concave and for minimization if the function $f(X)$ is a convex. As a result, the solution is determined by evaluating n equations and setting n derivatives partially to zero.

Mathematical programming is used when the objective and constraints of an optimization issue are expressed as mathematical functions and functional relations (Mirmohseni & Nasseri, 2017). A mathematical model whose requirements are expressed by linear relationships can be optimized using a technique called linear programming (LP, also known as linear optimization).

Ding et al. (2023) developed a novel approach to the problem of full fuzzy quadratic programming. The authors indicated that the issue with programming with fuzzy variables and coefficients was increasingly widespread. The A-PSO algorithm-based quadratic programming problem, with all of its parameters being fuzzy integers, was the subject of the study. In particular, the four triangle fuzzy number operations were expanded, and a better triangular fuzzy number sorting algorithm was suggested that took into account the different membership of each point. Furthermore, the innovative approach suggested in the paper provided the precise solution steps for the fully fuzzy quadratic programming problem. To evaluate and analyse the algorithm and outcomes, numerical examples were provided, which showed the efficiency and effectiveness of the suggested approach.

Forrester and Hunt-Isaak (2020) revisited two of the most popular linearization techniques for dealing with binary quadratic programmes in their study on the computational comparison of exact solution techniques for 0–1 quadratic programmes. They also looked at the potential for improving the formulations that had been proposed in the literature. They carried out a thorough computational analysis for five classes of binary quadratic programmes and contrasted the two approaches using a more recent linear reformulation of nonlinear programming to solver optimisation.

Gu and Chen (2020) developed the fundamental algorithm for the k-diagonal matrix-based zero-one unconstrained quadratic programming problem. They referred to it as the k-diagonal matrix zero-one unconstrained quadratic programming issue in their research. They suggested a Q matrix as a solution to such kind of issue. A derivation was performed to demonstrate the technique's viability. The new method was evaluated on a large number of numerical instances, and it was found that the procedure was simple to comprehend. By altering the value of k, the method's computation speed was examined, and it was found that the power of computation made the suggested method effective for issues with high dimensions.

Separable Quadratic Constrained Programming Problem being a problem with constraints has existing methods of solving it, such as the Wolf's Modified Simplex, Piecewise Linear Function, Beale's and Kuhn-Tucker Conditions. However, this problem can be handled by the introduction of the ordinary least square technique. It is in this regard, the study tends to demonstrate explicit algorithms to tackle such problems.

Description of the Proposed Approach for Separable Quadratic Programming Problem Mathematical programming

An optimization problem is said to be a Separable Quadratic Programming Problem (SQPP) if the objective function is quadratic, separable and concave whereas the constraints are linear (Opara & Isobeye, 2021). Since $Max[f(x)] = Min[-f(x)]$, the study concentrated on minimization problems. Hence, a separable quadratic programming problem is defined mathematically as

$$Min\ f(x) = \underline{c}^T X + X^T D X$$
$$\text{Subject to: } AX \geq b \tag{1}$$
$$X \geq 0$$

Where $\underline{c}^T = (c_1, c_2, c_3, \dots, c_n)$

$X = (x_1, x_2, x_3, \dots, x_n)^T$

$\underline{b} = (b_1, b_2, b_3, \dots, b_m)^T$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

The function $X^T D X$ defines a quadratic form, where D is a symmetric and positive definite matrix (Taha, 2007).

Linear Regression Model

The multiple linear regression has its basic model as:

$$Y_i = X_0 + X_1 Z_{i1} + X_2 Z_{i2} + \cdots + X_p Z_{ip} + \varepsilon_i \tag{2}$$

For each observation, $i = 1,2,3,\cdots,n$

Equation (2) considered n observations of one dependent (response) variable Y_i and p independent (explanatory) variables Z_{ij}. Hence, Y_i jth the observation of the dependent variable Z_{ij} is the ith observation of the jth independent variable. The value X_j ($j = 1,2,3,\cdots,p$) represents the parameters to be estimated, together with X_0 (intercept), and ε_i is the ith independent identically distributed normal error (Opara & Isobeye, 2021).

From Equation (2) for $i = 1,2,3,\cdots,n$, we have:

$$\left.\begin{aligned} Y_1 &= X_0 + X_1 Z_{11} + X_2 Z_{12} + \cdots + X_p Z_{1p} + \varepsilon_1 \\ Y_2 &= X_0 + X_1 Z_{21} + X_2 Z_{22} + \cdots + X_p Z_{2p} + \varepsilon_2 \\ Y_3 &= X_0 + X_1 Z_{31} + X_2 Z_{32} + \cdots + X_p Z_{3p} + \varepsilon_3 \\ &\vdots \\ Y_n &= X_0 + X_1 Z_{n1} + X_2 Z_{n2} + \cdots + X_p Z_{np} + \varepsilon_n \end{aligned}\right\} \tag{3}$$

Equation (3) in matrix form, is written as:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & Z_{11} & Z_{12} & \cdots & Z_{1p} \\ 1 & Z_{21} & Z_{22} & \cdots & Z_{2p} \\ 1 & Z_{31} & Z_{32} & \cdots & Z_{3p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & Z_{n1} & Z_{n2} & \cdots & Z_{np} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{pmatrix},$$

which when using matrix notation gives

$$Y = ZX + \varepsilon \tag{4}$$

where $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix}$, $Z = \begin{pmatrix} 1 & Z_{11} & Z_{12} & \cdots & Z_{1p} \\ 1 & Z_{21} & Z_{22} & \cdots & Z_{2p} \\ 1 & Z_{31} & Z_{32} & \cdots & Z_{3p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & Z_{n1} & Z_{n2} & \cdots & Z_{np} \end{pmatrix}$, $X = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$, $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{pmatrix}$

With non-negative parameters, the problem of the multiple linear regression model as in Equation (4) is stated as;

$$\left.\begin{aligned} Y &= ZX + \varepsilon \\ Subject \quad to \quad X &\geq 0 \end{aligned}\right\} \tag{5}$$

where Y is the vector of responses, Z is a $n \times p$ matrix, X is the unknown parameters of the model and $X \geq 0$ implies that all the elements in the column vector are non-negative, ε is the random error term of the linear regression model.

Incorporating inequality constraints and non-negative parameters in Equation (5), the general linear regression model can be written as;

$$\left.\begin{array}{c} Y = ZX + \varepsilon \\ Subject \quad to: \quad AX \geq b \\ X \geq 0 \end{array}\right\} \qquad (6)$$

where X is the unknown vector; $Z_{n\times p}(n \geq p)$ and $A_{m\times p}(m \leq p)$ are constant matrices, $Y_{n\times 1}$, $X_{p\times 1}$, $b_{m\times 1}$ and $\varepsilon_{n\times 1}$ are column vectors, $\varepsilon \sim N(0, \sigma^2 I)$, $Z^T Z \geq 0$ and $rank(A) = m$.

From Equation (4),

$$\varepsilon = Y - ZX \qquad (7)$$

Error sum of squares is;

$$\varepsilon^T \varepsilon = (Y - ZX)^T (Y - ZX) = Y^T Y - 2X^T Z^T Y + X^T Z^T ZX \qquad (8)$$

Hence, the general linear model in a regression analysis form with linear constraints can be written as;

$$\left.\begin{array}{l} Min. f_0(X) = (Y - ZX)^T (Y - ZX) \\ \quad Subject \quad to: \quad AX \geq b \\ \qquad\quad X \geq 0 \end{array}\right\} \qquad (9)$$

Comparing Equations (1) and (9), we have that

$$D = Z^T Z, \ \underline{c}^T = -2Y^T Z \text{ and } Y^T Y = 0$$

Employing the Maximum Likelihood Method, to estimate X in Equation (4), we recall that; if $x \sim N(\mu, \sigma^2)$, then;

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-\mu)^2}{2\sigma^2} \qquad (10)$$

Since $\varepsilon \sim N(0, \sigma^2 I)$ for linear regression model, then

$$f_0(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{\varepsilon^2}{2\sigma^2} \qquad (11)$$

The likelihood function is given by

$$L(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n; X, \sigma^2) = \prod_{i=1}^{n} f_0(\varepsilon_i)$$

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$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{\varepsilon_i^2}{2\sigma^2} = \frac{1}{(\sigma^2)^{\frac{n}{2}}(2\pi)^{\frac{n}{2}}} \exp - \frac{\sum_{i=1}^{n}\varepsilon_i^2}{2\sigma^2}$$

Since $\varepsilon = Y - ZX$, $\sum_{i=1}^{n}\varepsilon_i^2 = \varepsilon^T\varepsilon = (Y-ZX)^T(Y-ZX)$. Substitution gives

$$L(\varepsilon_1,\varepsilon_2,...,\varepsilon_n;X,\sigma^2) = \frac{1}{(\sigma^2)^{\frac{n}{2}}(2\pi)^{\frac{n}{2}}} \exp - \frac{(Y-ZX)^T(Y-ZX)}{2\sigma^2}$$

$$L(\varepsilon_1,\varepsilon_2,...,\varepsilon_n;X,\sigma^2) = (\sigma^2)^{-\frac{n}{2}}(2\pi)^{-\frac{n}{2}} \exp - \frac{(Y-ZX)^T(Y-ZX)}{2\sigma^2}$$

Taking the natural logarithm of both sides and simplifying, give

$$\ln L(\varepsilon_1,\varepsilon_2,...,\varepsilon_n;X,\sigma^2) = -\frac{n}{2}\ln(\sigma^2) - \frac{n}{2}\ln(2\pi) - \frac{(Y-ZX)^T(Y-ZX)}{2\sigma^2}$$

To maximize $\ln(L)$, we differentiate with respect to X , set the derivative equal to zero, and solve for X .

But

$$(Y-ZX)^T(Y-ZX) = (Y^T - X^TZ^T)(Y-ZX) = Y^TY - Y^TZX - X^TZ^TY + X^TZ^TZX$$
$$= Y^TY - 2X^TZ^TY + X^TZ^TZX$$

$$\therefore \ln L(\varepsilon_1,\varepsilon_2,...,\varepsilon_n;X,\sigma^2) = -\frac{n}{2}\ln(\sigma^2) - \frac{n}{2}\ln(2\pi) - \frac{Y^TY - 2X^TZ^TY + X^TZ^TZX}{2\sigma^2} \quad \text{and}$$

$$\frac{\partial \ln L}{\partial X} = \frac{-2Z^TY + 2(Z^TZ)X}{2\sigma^2}$$

since the derivative of the first two terms of $\ln(L)$ is zero.

To minimize, the equation

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$$\frac{-2Z^TY+2(Z^TZ)X}{2\sigma^2}=0$$

$$-2Z^TY+2(Z^TZ)X=0$$

$$\Rightarrow 2(Z^TZ)X=2Z^TY$$

$$\therefore X=\frac{Z^TY}{(Z^TZ)}=(Z^TZ)^{-1}Z^TY \quad (12)$$

if $D=Z^TZ$, then

$$X=D^{-1}Z^TY \quad (13)$$

To estimate X in the objective function of Equation (1), we have

$$\frac{\partial f(X)}{\partial X}=\underline{c}^T+2X^TD=0$$

$$\underline{c}^T=-2X^TD$$

$$\therefore \underline{c}=-2D^TX$$

$$\Rightarrow \underline{c}=-2DX$$

Since D is a symmetric matrix

$$\therefore X=-\frac{\underline{c}}{2D} \quad (14)$$

Theorem 1: The association between the vector $Y_{n\times 1}$, and the matrix $Z_{n\times p}$ is given by

$$Y=-\frac{1}{2}Z^{-1}\underline{c}\text{; where } Z=D^{\frac{1}{2}}$$

Proof

Equate Equations (12) and (14) to get;

$$(Z^TZ)^{-1}Z^TY=-\frac{\underline{c}}{2D}$$

$$2D(Z^TZ)^{-1}Z^TY=-\underline{c}$$

$$2(Z^TZ)(Z^TZ)^{-1}Z^TY=-\underline{c} \quad (15)$$

Where $(Z^TZ)=D$ and

$$Z = D^{\frac{1}{2}} \tag{16}$$

From Equation (15), we have

$$2Z^T Y = -\underline{c}$$

$$2(Z^T)^{-1} Z^T Y = -(Z^T)^{-1}\underline{c}$$

$$2Y = -(Z^T)^{-1}\underline{c}$$

$$Y = -\frac{1}{2}(Z^T)^{-1}\underline{c}$$

$$\therefore Y = -\frac{1}{2}Z^{-1}\underline{c} \tag{17}$$

Since Z is a symmetric matrix

Theorem 2: The analogy of SQPP and GLM is obtained as

$$f_0(X) = f(X) + \frac{1}{4}c^T D^{-1} c$$

Proof

$$f\ (X) = \underline{c}^T X + X^T DX \text{ and } f_0(X) = Y^T Y - 2X^T Z^T Y + X^T Z^T ZX$$

$$f_0(X) - Y^T Y + 2X^T Z^T Y - X^T Z^T ZX = f\ (X) - \underline{c}^T X - X^T DX$$

$$f_0(X) - Y^T Y + 2X^T Z^T Y - X^T DX = f\ (X) - \underline{c}^T X - X^T DX$$

$$= f\ (X) - \underline{c}^T X + Y^T Y - 2X^T Z^T Y$$

$$= f\ (X) - \underline{c}^T X + \left(-\frac{1}{2}Z^{-1}c\right)^T \left(-\frac{1}{2}Z^{-1}c\right) - 2X^T Z^T \left(-\frac{1}{2}Z^{-1}c\right)$$

$$= f\ (X) - \underline{c}^T X + \frac{1}{4}c^T \left(Z^{-1}\right)^T Z^{-1}c + X^T Z^T Z^{-1}c$$

$$= f\ (X) - \underline{c}^T X + \frac{1}{4}c^T D^{-\frac{1}{2}} D^{-\frac{1}{2}} c + X^T D^{\frac{1}{2}} D^{-\frac{1}{2}} c$$

$$= f\ (X) - \underline{c}^T X + \frac{1}{4}c^T D^{-1} c + X^T c$$

$$\therefore f_0(X) = f(X) + \frac{1}{4}c^T D^{-1} c$$

Hence, the algorithm for the proposed technique becomes

Step One

Solve for $X = D^{-1}Z^T Y$ and check if the conditions for $AX \geq \underline{b}$ or $AX \leq \underline{b}$ are satisfied. If they are satisfied, stop; the present solution is optimal, otherwise go to step two

Step Two

Solve for Minimize $f_0(X^1) = (Y - ZX^{(1)})^T (Y - ZX^{(1)})$ subject to any constraint (which contains the whole unknown variables) and check if the conditions for $AX^{(1)} \geq \underline{b}$ or $AX^{(1)} \leq \underline{b}$ are satisfied. If they are satisfied, stop; the present solution is optimal, otherwise, continue until you exhaust all the single constraint, and go to the next step if any of them did not provide optimal solution.

Step Three

Solve for Minimize $f_0(X^{(i,j)}) = (Y - ZX^{(i,j)})^T (Y - ZX^{(i,j)})$ subject to every pair of constraints and check if the conditions for $AX^{(i,j)} \geq \underline{b}$ or $AX^{(i,j)} \leq \underline{b}$ are satisfied. If they are satisfied, stop; the present solution is optimal, otherwise go to the next iteration. The process continues until an optimal solution is achieved.

Demonstration of the OLS Technique with Numerical Problem Sets

Problem Set One

Maximize $f(x_1, x_2) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

Subject to $\quad x_1 + 2x_2 \leq 4$

$\quad x_1, x_2 \geq 0$

(Source: Gupta; 2011)

Solution

Changing the problem to minimization, we have Minimize $f(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 + 3x_2^2$

$$A = \begin{pmatrix} 1 & 2 \end{pmatrix},\ b = (4),\ D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},\ c = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$$

Iteration One

Solve for $X = D^{-1}Z^T Y$

where

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$$Z = D^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

and

$$Y = -\frac{1}{2} Z^{-1} \underline{c} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ -6 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -4 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2\sqrt{3}} \end{pmatrix}\begin{pmatrix} -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore x_1 = 2, x_2 = 1$$

To check for optimality, we have $x_1 + 2x_2 = 2 + 2(1) = 4$. Since $x_1 + 2x_2 \leq 4$, we stop the present solution is optimal at iteration one. Therefore $x_1 = 2$ and $x_2 = 1$ with the objective value of $f = 4(2) + 6(1) - (2)^2 - 3(1)^2 = 7$

Problem Set Two

Two items, P and Q, are produced by a firm. One unit of product P takes 30 minutes to process, while one unit of product Q takes 15 minutes. 35 hours per week is the maximum machine time that can be used. Compared to product Q, which needs 3 kg of raw material for every unit, product P only needs 2 kg. Only 180 kg of raw materials are accessible each week. P and Q, two goods with limitless market potential, are sold for ₦ 200 and ₦ 500 each, respectively, per unit. If the production costs for goods A and B are $2x_1^2$ and $3x_2^2$ respectively, find how much of each product should be produced per week, where x_1 and x_2 are respectively the quantities of P and Q to be produced. (Extracted from Gupta and Hira, 2011)

If the production costs for goods A and B are the same

Solution

The first thing is to formulate the mathematical model; hence x_1 and x_2 are the quantities of products P and Q respectively, which are to be manufactured per week. The selling price of products P and Q is ₦ 200 and ₦ 500 per unit respectively.

Therefore, the total revenue per week = $200x_1 + 500x_2$

The manufacturing cost of P is $2x_1^2$ and that of Q is $3x_2^2$ per unit.

Thus; total manufacturing costs per week = $2x_1^2 + 3x_2^2$

Therefore, profit per week = $200x_1 + 500x_2 - 2x_1^2 - 3x_2^2$

The machining of product P requires 30 minutes per unit, while product Q requires 15 minutes per unit. Since a maximum of 35 hours of machining time are available,

$$30x_1 + 15x_2 \leq 35 \times 60$$

or $$2x_1 + x_2 \leq 140$$

The constraint on the availability of raw material is expressed as

$$2x_1 + 3x_2 \leq 180$$

$$x_1, x_2 \geq 0$$

In summary, the problem can be expressed as,

Maximize $f(x_1, x_2) = 200x_1 + 500x_2 - 2x_1^2 - 3x_2^2$

Subject to $2x_1 + x_2 \leq 140$

$$2x_1 + 3x_2 \leq 180$$

$$x_1, x_2 \geq 0$$

Here, the objective function is non-linear, while the constraints are linear.

Changing the problem to minimization, we have Minimize $f(x_1, x_2) = -200x_1 - 500x_2 + 2x_1^2 + 3x_2^2$

Putting the above NLPP in matrix form, we have

$$\text{Minimize } f(x_1, x_2) = \begin{pmatrix} -200 & -500 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Subject to } \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 140 \\ 180 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

Where

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix},\ b = \begin{pmatrix} 140 \\ 180 \end{pmatrix},\ D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},\ c = \begin{pmatrix} -200 \\ -500 \end{pmatrix}.$$

Step One

Solve for $X = D^{-1}Z^{T}Y$

where

$$Z = D^{\frac{1}{2}} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

and

$$Y = -\frac{1}{2}Z^{-1}\underline{c} = -\frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}^{-1}\begin{pmatrix} -200 \\ -500 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -200 \\ -500 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{3}} \end{pmatrix}\begin{pmatrix} -200 \\ -500 \end{pmatrix} = \begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 100 \\ 250 \end{pmatrix} = \begin{pmatrix} 50 \\ \frac{250}{3} \end{pmatrix}$$

$$\therefore x_1 = 50, x_2 = \frac{250}{3}$$

To check for optimality, we have $2x_1 + x_2 = 2(50) + \frac{250}{3} = \frac{550}{3} = 183.333 > 140$ and $2x_1 + 3x_2 = 2(50) + 3\left(\frac{250}{3}\right) = 350 > 180$. Since $2x_1 + x_2 \leq 140$ and $2x_1 + 3x_2 \leq 180$ conditions did not hold, we go to the next step.

Step Two

We solve with the first constraint given as:

Minimize $f_0(X^{(1)}) = (Y - ZX^{(1)})^T(Y - ZX^{(1)})$

Subject to: $2x_1^{(1)} + x_2^{(1)} = 140$

$x_1^{(1)}, x_2^{(1)} \geq 0$

Making $x_2^{(1)}$ the subject of the formula in the first constraint, we have

$$x_2^{(1)} = 140 - 2x_1^{(1)}$$

Substituting $x_2^{(1)}$, Y and Z in the objective function to obtain

$$f_0(X^{(1)}) = \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ 140 - 2x_1^{(1)} \end{pmatrix}\right]^T \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ 140 - 2x_1^{(1)} \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2}x_1^{(1)} \\ 140\sqrt{3} - 2\sqrt{3}x_1^{(1)} \end{pmatrix}\right]^T \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2}x_1^{(1)} \\ 140\sqrt{3} - 2\sqrt{3}x_1^{(1)} \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 50\sqrt{2} - \sqrt{2}x_1^{(1)} \\ -\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)} \end{pmatrix}\right]^T \left[\begin{pmatrix} 50\sqrt{2} - \sqrt{2}x_1^{(1)} \\ -\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)} \end{pmatrix}\right]$$

$$= \begin{pmatrix} 50\sqrt{2} - \sqrt{2}x_1^{(1)} & -\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)} \end{pmatrix} \begin{pmatrix} 50\sqrt{2} - \sqrt{2}x_1^{(1)} \\ -\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)} \end{pmatrix}$$

$$\therefore\ f_0(X^{(1)}) = \left(50\sqrt{2} - \sqrt{2}x_1^{(1)}\right)^2 + \left(-\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)}\right)^2$$

$$\frac{\partial f_0(X^{(1)})}{\partial x_1^{(1)}} = -2\left(50\sqrt{2} - \sqrt{2}x_1^{(1)}\right)(\sqrt{2}) + 2\left(-\frac{170\sqrt{3}}{3} + 2\sqrt{3}x_1^{(1)}\right)(2\sqrt{3}) = 0$$

$$-200 + 4x_1^{(1)} - 680 + 24x_1^{(1)} = 0$$

$$28x_1^{(1)} = 880 \Rightarrow x_1^{(1)} = \frac{220}{7}$$

$$\therefore\ x_2^{(1)} = 140 - 2\left(\frac{220}{7}\right) = \frac{540}{7}$$

To check for optimality, we have $2x_1 + x_2 = 2\left(\frac{220}{7}\right) + \frac{540}{7} = 140 = 140$ and $2x_1 + 3x_2 = 2\left(\frac{220}{7}\right) + 3\left(\frac{540}{7}\right) = \frac{2060}{7} = 294.286 > 180$. Since the result does not satisfy $2x_1 + 3x_2 \le 180$ the condition, we go to the next step.

Step Three

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We solve with the second constraint given as:

Minimize $f_0(X^{(2)}) = (Y - ZX^{(2)})^T (Y - ZX^{(2)})$

Subject to: $2x_1^{(2)} + 3x_2^{(2)} = 180$

$x_1^{(2)}, x_2^{(2)} \geq 0$

Making $x_1^{(2)}$ the subject of the formula in the second constraint, we have

$$x_1^{(2)} = 90 - 1.5x_2^{(2)}$$

Substituting $x_1^{(2)}$, Y and Z in the objective function to obtain

$$f_0(X^{(2)}) = \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 90 - 1.5x_2^{(2)} \\ x_2^{(2)} \end{pmatrix}\right]^T \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 90 - 1.5x_2^{(2)} \\ x_2^{(2)} \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} 90\sqrt{2} - \frac{3\sqrt{2}x_2^{(2)}}{2} \\ \sqrt{3}x_2^{(2)} \end{pmatrix}\right]^T \left[\begin{pmatrix} 50\sqrt{2} \\ \frac{250\sqrt{3}}{3} \end{pmatrix} - \begin{pmatrix} 90\sqrt{2} - \frac{3\sqrt{2}x_2^{(2)}}{2} \\ \sqrt{3}x_2^{(2)} \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} -40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2} \\ \frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)} \end{pmatrix}\right]^T \left[\begin{pmatrix} -40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2} \\ \frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)} \end{pmatrix}\right]$$

$$= \begin{pmatrix} -40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2} & \frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)} \end{pmatrix} \begin{pmatrix} -40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2} \\ \frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)} \end{pmatrix}$$

$$\therefore\ f_0(X^{(2)}) = \left(-40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2}\right)^2 + \left(\frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)}\right)^2$$

$$\frac{\partial f_0(X^{(2)})}{\partial x_2^{(2)}} = 2\left(-40\sqrt{2} + \frac{3\sqrt{2}x_2^{(2)}}{2}\right)\frac{3\sqrt{2}}{2} - 2\left(\frac{250\sqrt{3}}{3} - \sqrt{3}x_2^{(2)}\right)(\sqrt{3}) = 0$$

$$-240 + 9x_2^{(2)} - 500 + 6x_2^{(2)} = 0$$

$$15x_2^{(2)} = 740 \Rightarrow x_2^{(2)} = \frac{740}{15} = \frac{148}{3}$$

$$\therefore x_1^{(2)} = 90 - 1.5\left(\frac{148}{3}\right) = 16$$

To check for optimality, we have $2x_1 + x_2 = 2(16) + \frac{148}{3} = \frac{244}{3} = 81.333 < 140$ and

$2x_1 + 3x_2 = 2(16) + 3\left(\frac{148}{3}\right) = 180 = 180$. Since $2x_1 + x_2 \le 140$ and $2x_1 + 3x_2 \le 180$ conditions are met, we stop. The present solution is optimal. Therefore $x_1 = 16$ and $x_2 = \frac{148}{3}$ with the objective value of $f = 200(16) + 500\left(\frac{148}{3}\right) - 2(16)^2 - 3\left(\frac{148}{3}\right)^2 = 20053.3333$

Conclusion

A new approach of solving a Separable Quadratic Constrained Programming Problem is developed. Two examples have been tackled to confirm the efficiency of the developed technique. The results obtained from Wolfram Mathematica software agreed with the result obtained via the proposed technique. It can be concluded that the algorithms of solving a separable quadratic constrained programming problem stated in this study are quite explicit, efficient and very easy to employ.

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